CBC Parameter Estimation

GW Open Data Workshop #7, 2024

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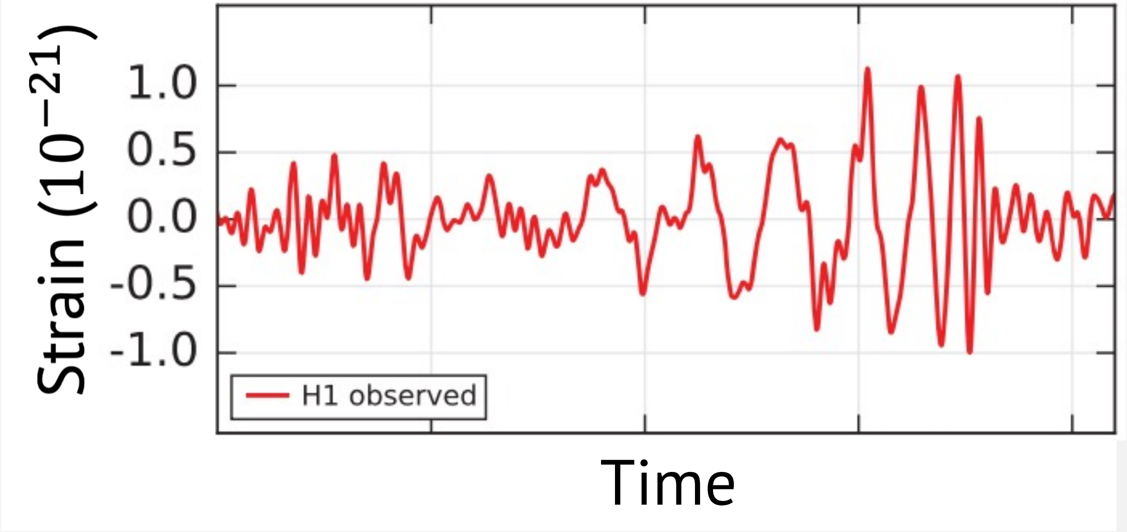
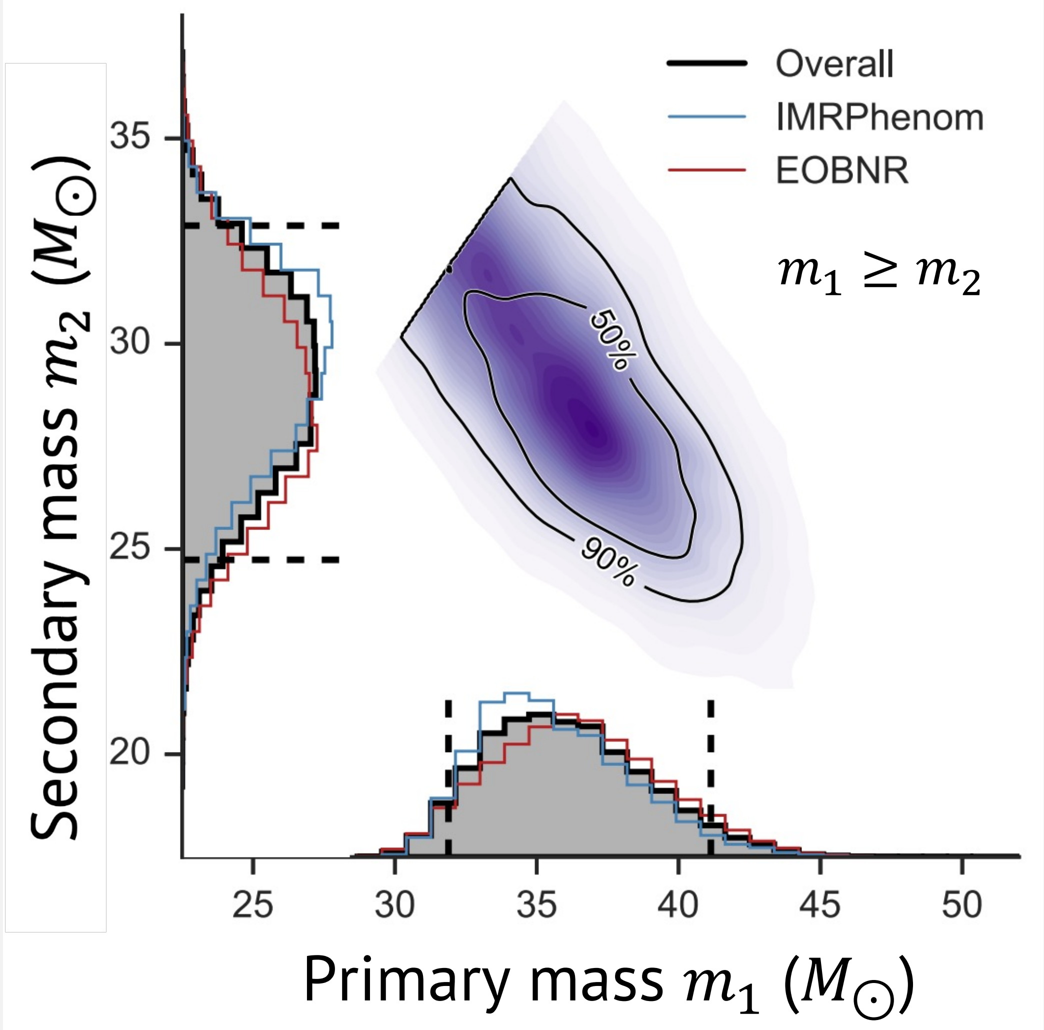
Source Characterization from Data

Credit:
B. P. Abbott et al., PRL **116**, no.6, 061102 (2016).



Credit: B. P. Abbott et al., PRL **116**, no.6, 241102 (2016).



Masses: m_1, m_2

Higher masses
→ Shorter and louder signal

Chirp mass $\mathcal{M}$ is measured most precisely,

$$\mathcal{M} = \frac{(m_1 m_2)^{\frac{3}{5}}}{(m_1 + m_2)^{\frac{1}{5}}}.$$

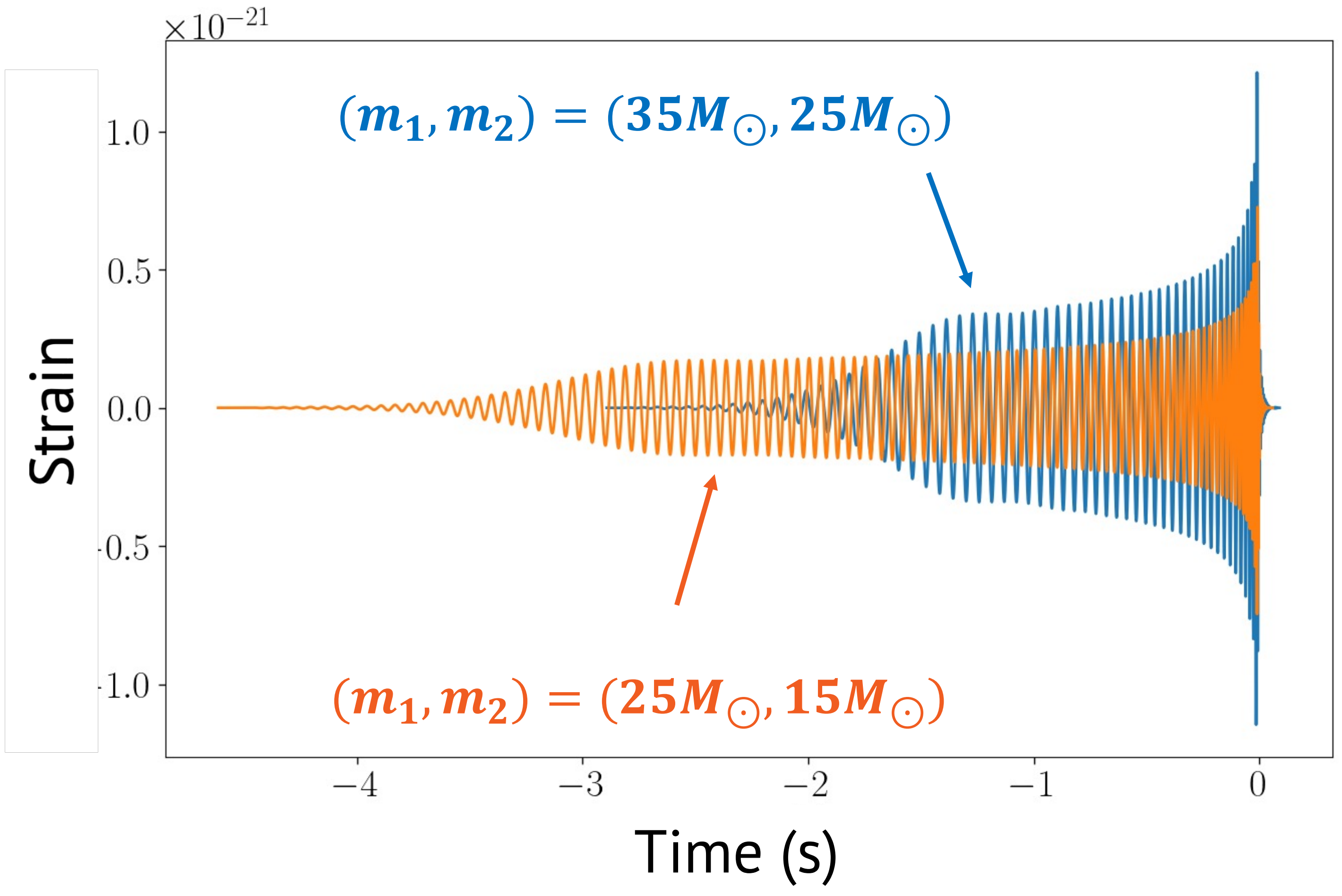
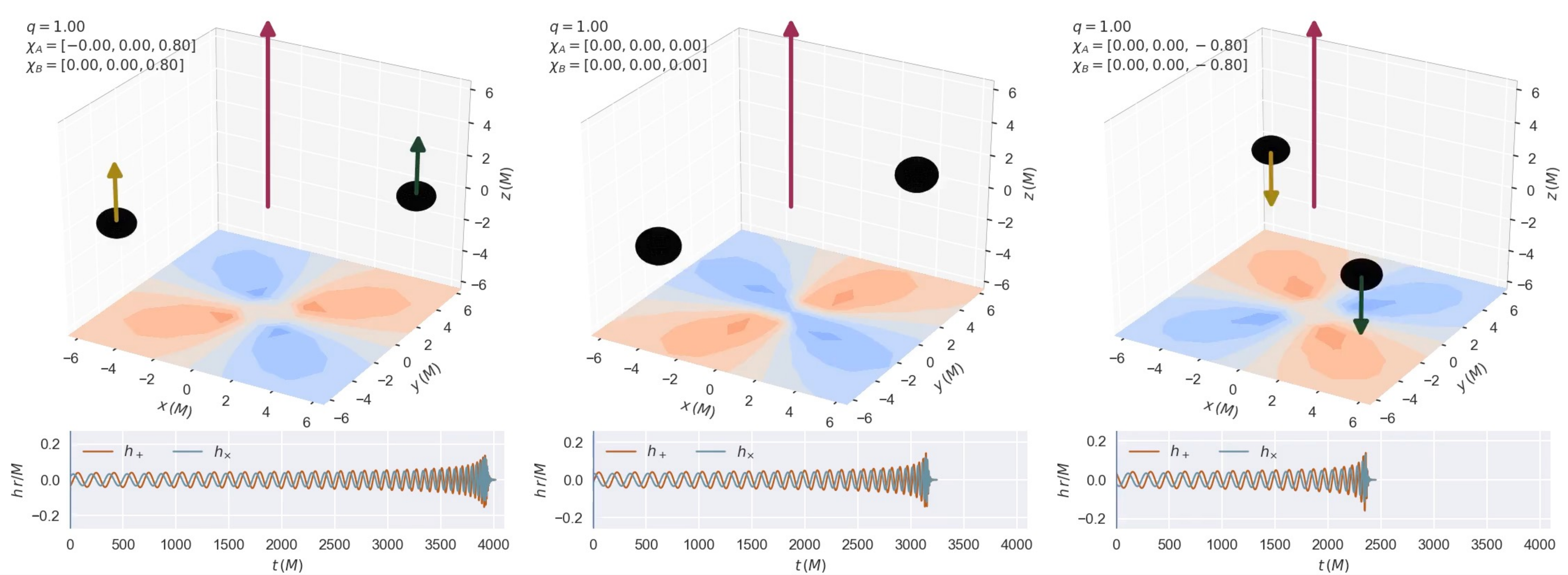


Figure: CBC signals starting from 20Hz

Spins: $\overrightarrow{\chi_1}, \overrightarrow{\chi_2}$

Spins aligned with orbital angular momentum → longer signal



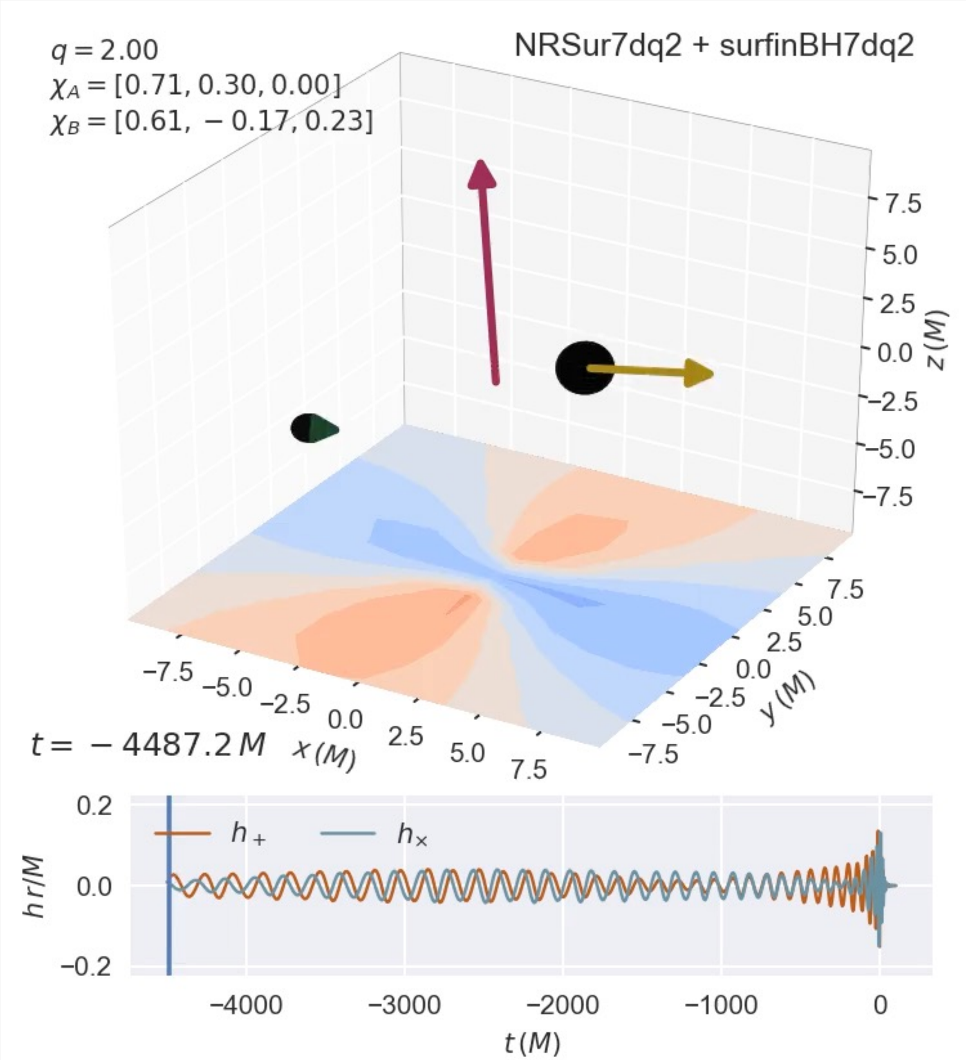
Credit: Vijay Varma et al., Binary Black Hole Explorer

Spins: $\overrightarrow{\chi_1}, \overrightarrow{\chi_2}$

Orthogonal spin components

→ Precession of orbital plane

→ Amplitude and phase modulation

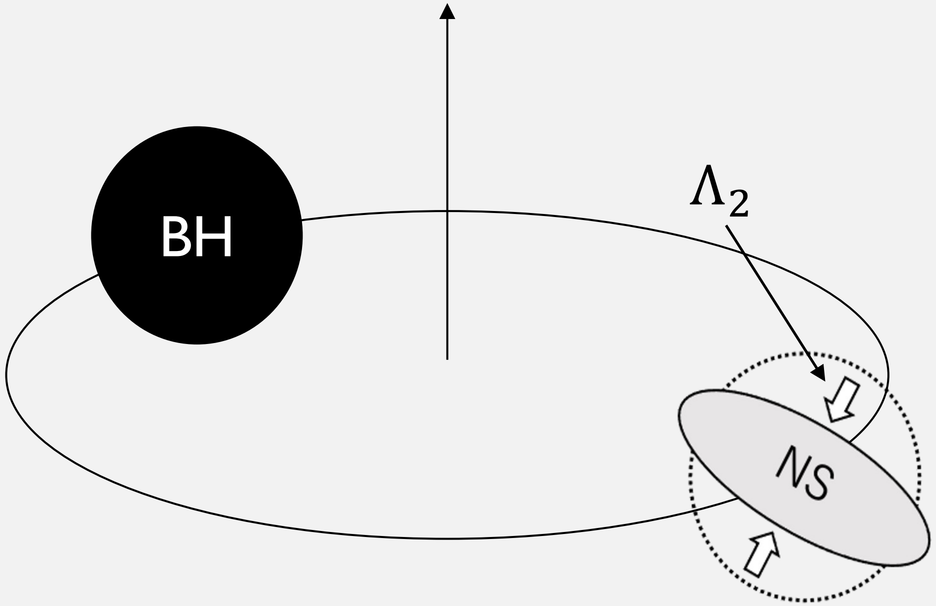


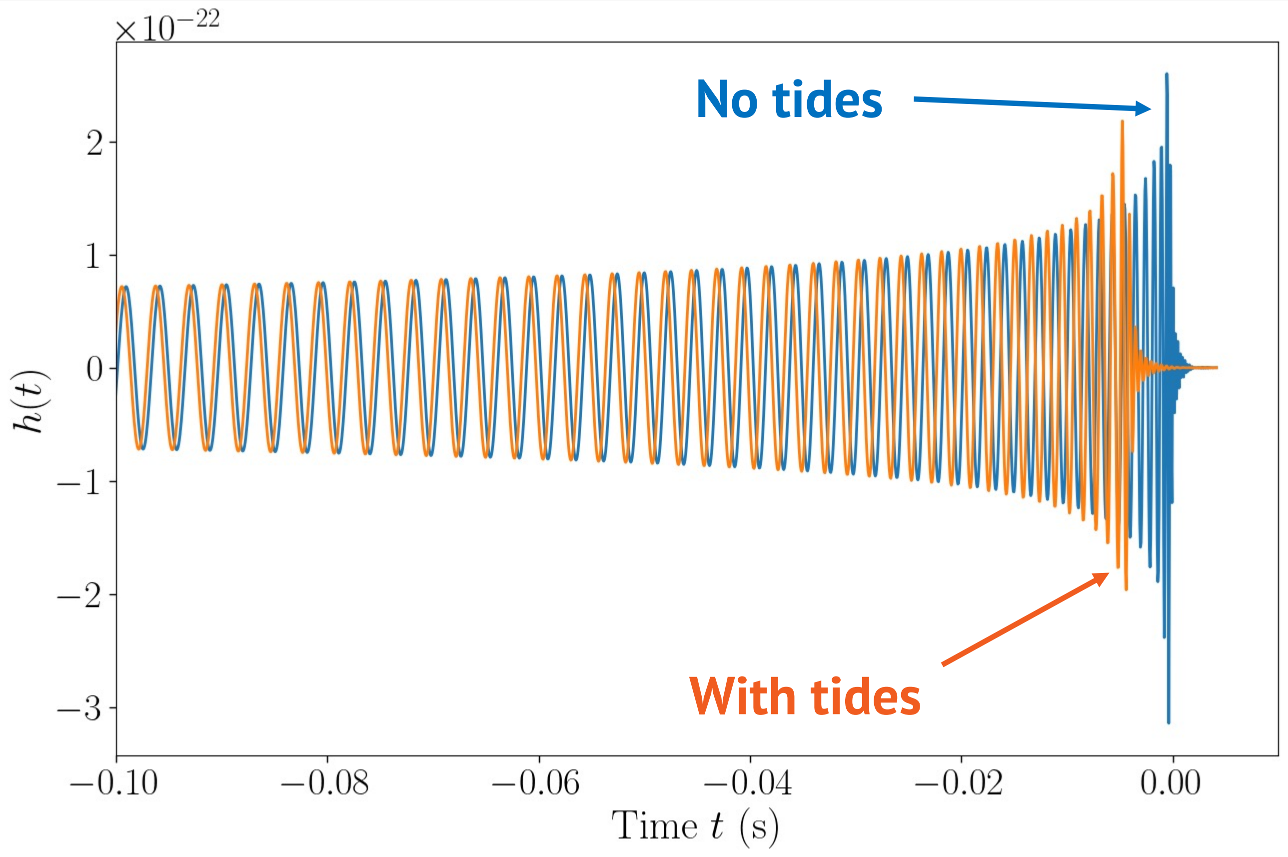
Masses and spins are key to probe the formation history of merging binary black holes.

Credit: Vijay Varma et al., Binary Black Hole Explorer

Tidal deformability parameters: Λ_1, Λ_2

Tidal deformation of star accelerates orbital motion.

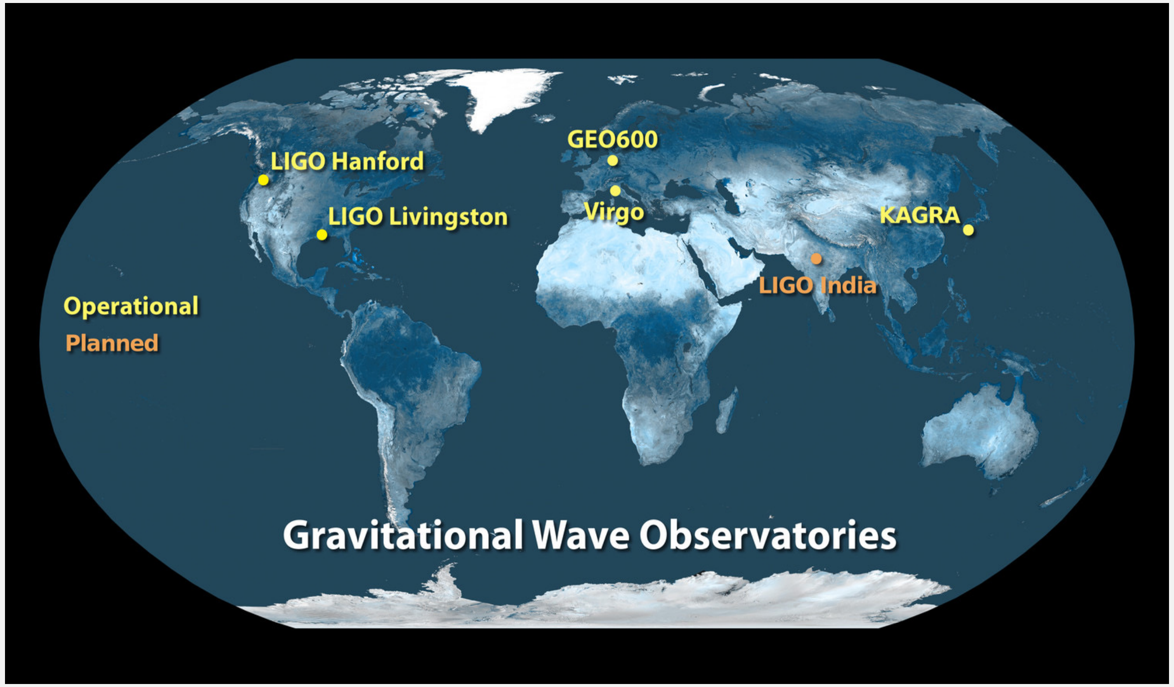




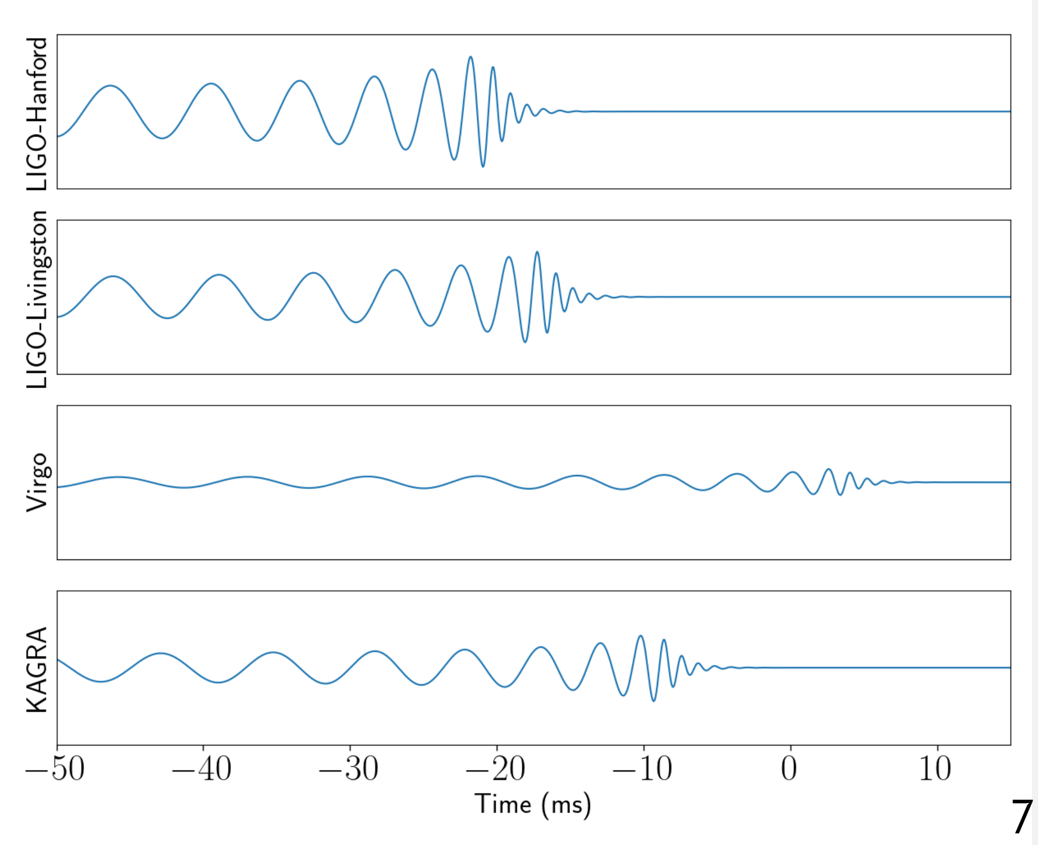
Can constrain the properties of highly dense matter.

Source direction

Gravitational waves



Source direction is estimated with arrival time, amplitudes, and phases observed at multiple detectors.



Source parameters characterizing signal

15 binary black hole parameters + **1** additional parameter **per neutron star**

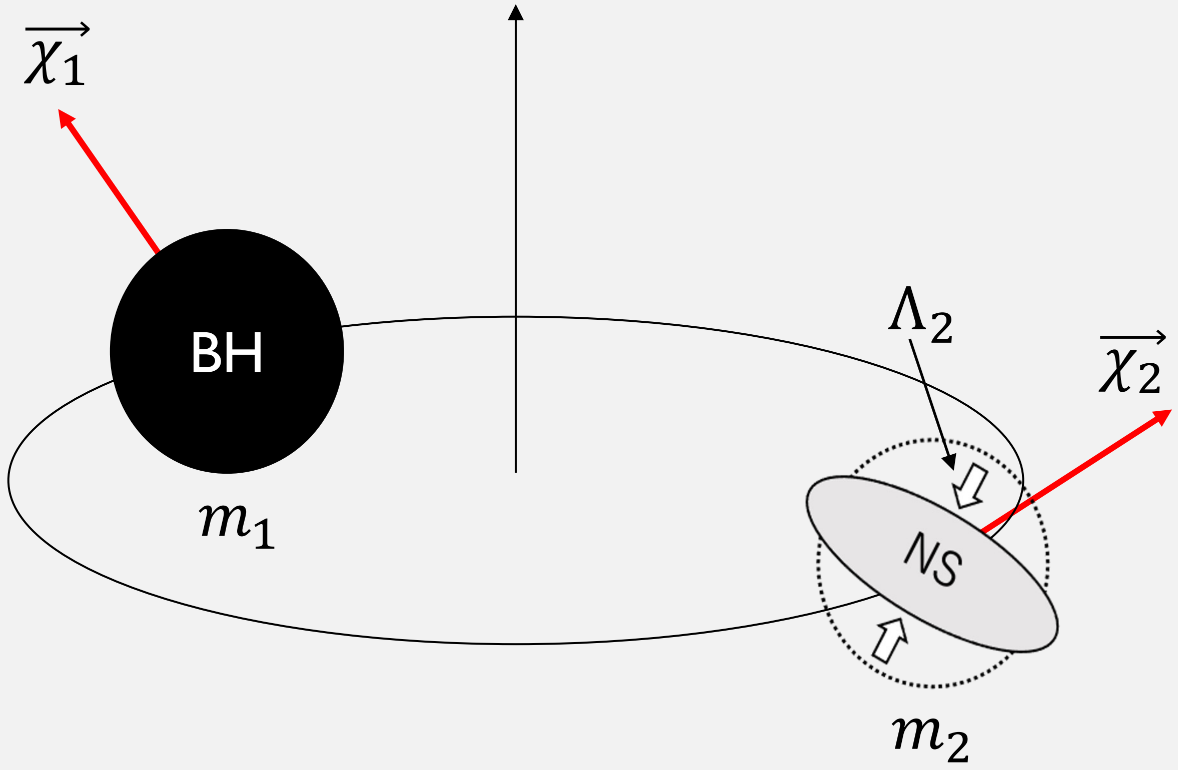


Figure: Schematic picture of neutron star black hole

- Masses: m_1, m_2
  (Chirp mass $\mathcal{M}$ and mass ratio $q \equiv m_2/m_1$ used for efficiency)
- Spins: $\overrightarrow{\chi_1}, \overrightarrow{\chi_2}$
  (Spin magnitudes and angles typically used)
- Tidal deformabilities: Λ_1, Λ_2
  (only for neutron stars)
- Right ascension RA/declination Dec
- Coalescence time t_c
  (Detector frame sky coordinates and time often used for efficiency)
- Luminosity distance D_L
- Orbital inclination angle θ_{JN}
- Polarization angle ψ
- Coalescence phase ϕ_c

Calibration uncertainties

L. Sun *et al.*, arXiv: 2107.00129.

Due to uncertainties in detector calibration, observed signal can be slightly different from true signal:

$$\tilde{h}_{\mathrm{observed}}(f) = \tilde{h}_{\mathrm{true}}(f)\big(1 + \delta A(f)\big)e^{i\delta\phi(f)}.$$

Additional $2N_{\mathrm{nodes}}$ parameters per detector: $\{\delta A(f_i), \delta\phi(f_i)\}\quad (i = 1,2,\ldots,N_{\mathrm{nodes}})$.

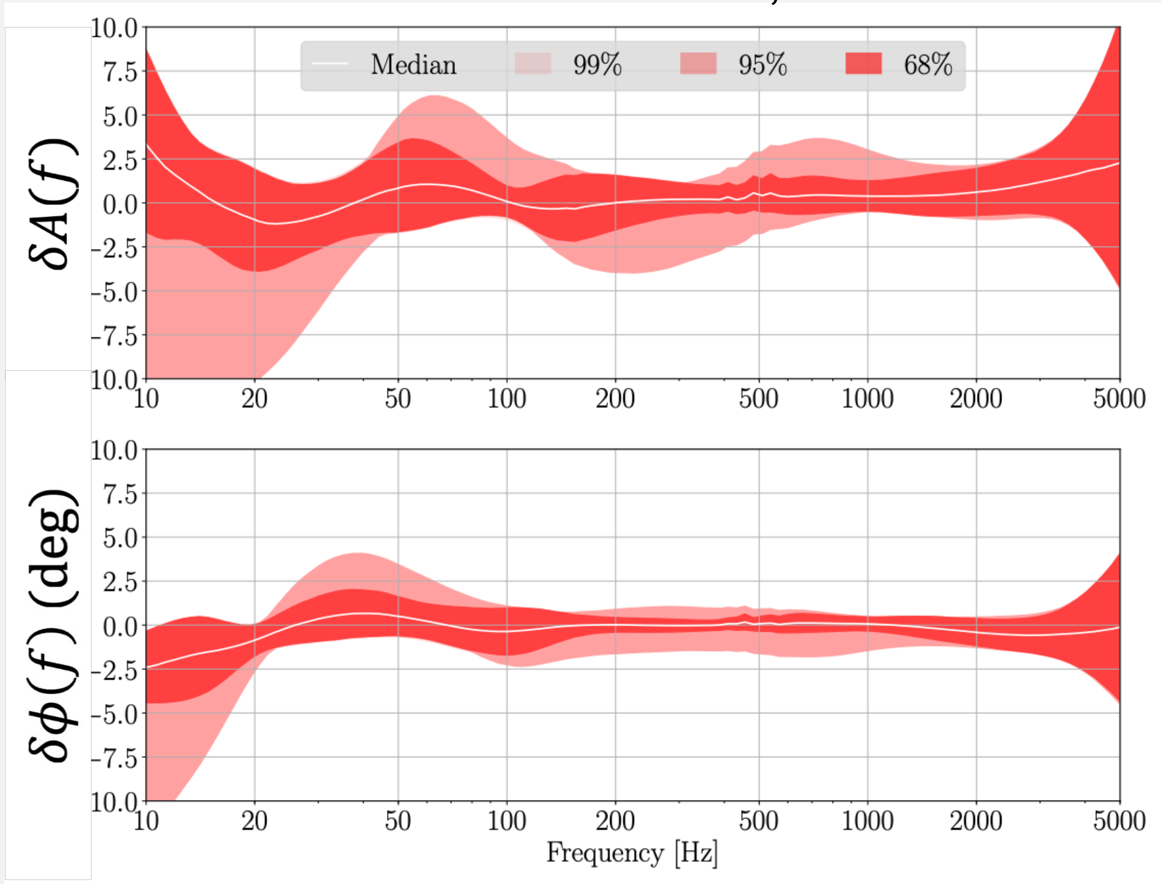


Figure: Calibration uncertainties of amplitude (top) and phase (bottom) of LIGO-Hanford in O3

Bayesian inference

Likelihood — Prior

Posterior → $$p(\theta|d,M) = \frac{p(d|\theta,M)p(\theta|M)}{p(d|M)}$$ ← Evidence

d: observed data

θ: parameters (masses, spins etc.)

M: Signal hypothesis

Bayesian inference

Likelihood ↓ Prior ↓

Posterior → $$p(\theta|d,M) = \frac{\mathcal{L}(d|\theta,M)\pi(\theta|M)}{\mathcal{Z}(d|M)}$$ ← Evidence

d: observed data

θ: parameters (masses, spins etc.)

M: Signal hypothesis

Bayesian inference

$$\underbrace{p(\theta|d,M)}_{\text{Posterior}} = \frac{\overbrace{\mathcal{L}(d|\theta,M)}^{\text{Likelihood}}\,\overbrace{\pi(\theta|M)}^{\text{Prior}}}{\underbrace{\mathcal{Z}(d|M)}_{\text{Evidence}}}$$

Prior encodes our prior knowledge or belief on θ.

- No information available → Use uninformative prior (e.g. isotropic on RA/Dec, uniform in masses etc.).
- It can incorporate information from electromagnetic observations or astrophysics (e.g. fixed to RA/Dec from electromagnetic obs., astrophysical mass prior etc.).

Bayesian inference

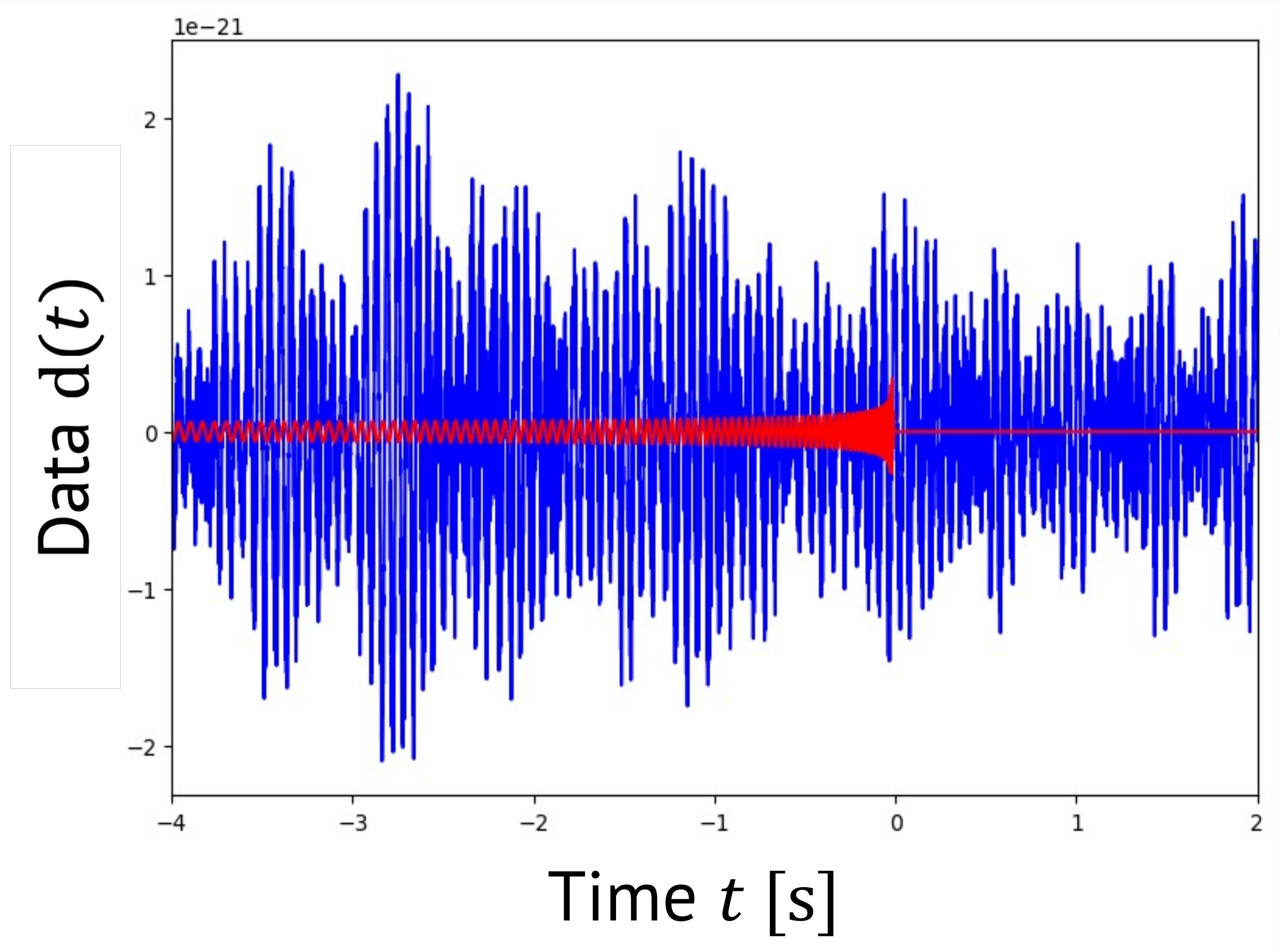
$$\text{Posterior} \rightarrow p(\theta|d,M) = \frac{\overbrace{\mathcal{L}(d|\theta,M)}^{\text{Likelihood}}\overbrace{\pi(\theta|M)}^{\text{Prior}}}{\mathcal{Z}(d|M)} \quad \leftarrow \text{Evidence}$$

Evidence can be used for comparing different hypotheses/models (e.g. noise vs signal hypotheses, different waveform models etc.).

$$B = \frac{\mathcal{Z}(d|M_1)}{\mathcal{Z}(d|M_2)}, \qquad B \gg 1 \rightarrow M_1 \text{ is favored}, \qquad B \ll 1 \rightarrow M_2 \text{ is favored.}$$

M_1, M_2: two different hypotheses/models

Likelihood: $\mathcal{L}(d|\theta, M)$



CBC signal Noise

$$d(t) = h(t;\theta) + n(t).$$

$$\mathcal{L}(d|\theta, M) = p(d - h(\theta)|\text{Noise})$$

Likelihood: $\mathcal{L}(d|\theta, M)$

- Noise is weakly stationary: $\langle n(t)\rangle = \mathrm{const.}, \langle n(t)n(t')\rangle = R(|t-t'|)$.

$$\longrightarrow \langle \tilde{n}(f_l)\rangle = 0 \left(f_l = \frac{l}{T} > 0, T\text{: data duration}\right), \langle \tilde{n}^*(f_l)\tilde{n}(f_{l'})\rangle \simeq \frac{TS(f_l)}{2}\delta_{ll'}.$$

$S(f_l) = 2\langle|\tilde{n}(f_l)|^2\rangle/T$ is referred to as Power Spectral Density (PSD) and characterizes noise variance at f_l.

- Noise follows Gaussian distribution.

Likelihood: $\mathcal{L}(d|\theta, M)$

Those assumptions lead to Whittle likelihood,

$$p(n|\text{Noise}) = \exp\left(-\frac{2}{T}\sum_l \frac{|\tilde{n}(f_l)|^2}{S(f_l)}\right).$$

$$\longrightarrow \mathcal{L}(d|\theta, M) \propto \exp\left[-\frac{2}{T}\sum_l \frac{\left|\tilde{d}(f_l) - \tilde{h}(f_l;\theta)\right|^2}{S(f_l)}\right].$$

Higher likelihood → Smaller residual $\left|\tilde{d}(f_l) - \tilde{h}(f_l;\theta)\right|$

See J. Veitch et al. (2015): https://arxiv.org/abs/1409.7215 for more context.

PSD estimation

- Average tens-hundreds of data sets which do not contain signal: $S(f_l) = 2\langle|\tilde{n}(f_l)|^2\rangle/T$.
- Fit the spectra of on-source data to mitigate biases from non-stationary noise (See Littenberg and Cornish (2015): https://arxiv.org/abs/1410.3852).

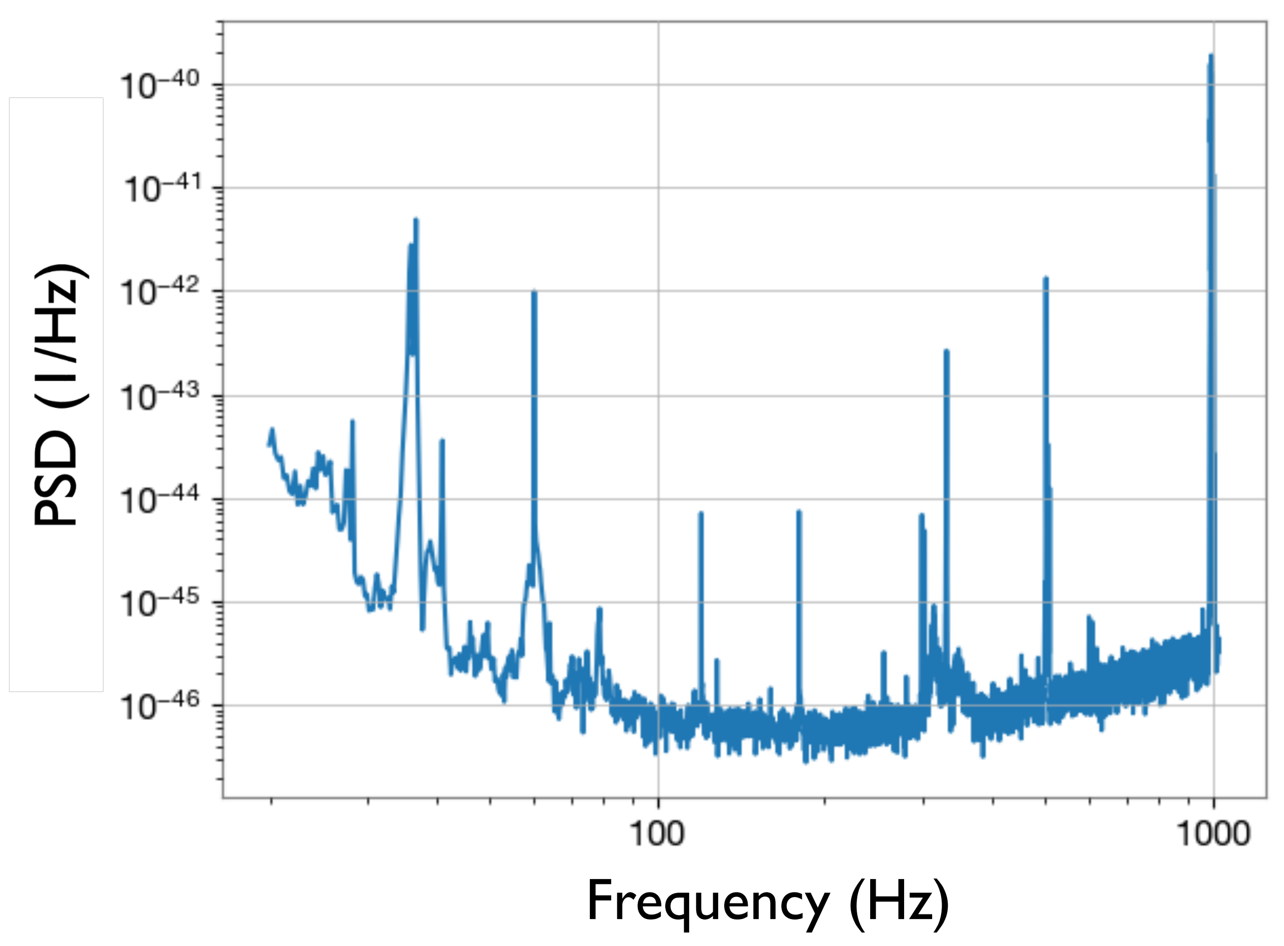


Figure: PSD estimated from data around GW150914

Marginalization

- 1D posterior distribution

$$p(m_2|d,M) = \int p(\theta|d,M)\underline{dm_1 d\overrightarrow{\chi_1} d\overrightarrow{\chi_2} \ldots}$$

Except for m_2

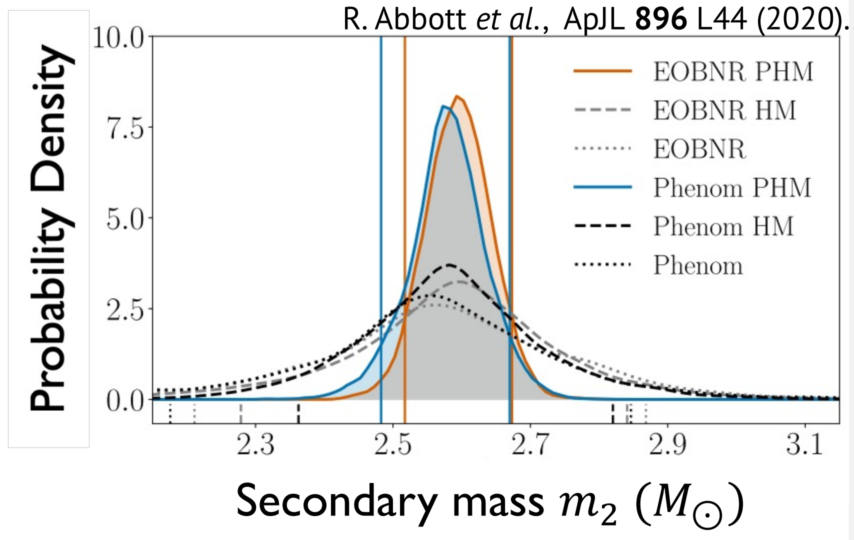
- 2D posterior distribution

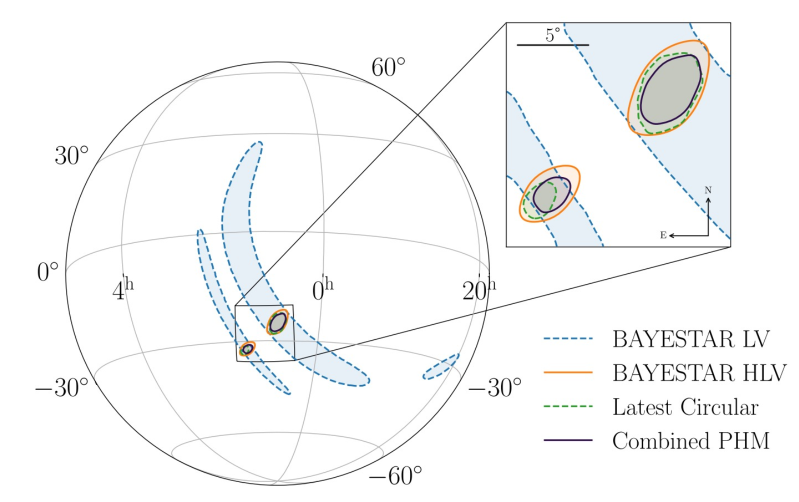
$$p(\mathrm{RA},\mathrm{Dec}|d,M) = \int p(\theta|d,M)\underline{dm_1 dm_2 d\overrightarrow{\chi_1} d\overrightarrow{\chi_2} \ldots}$$

Except for RA, Dec

They require high-dimensional numerical integration.

Figure credit:
R. Abbott *et al.*, ApJL **896** L44 (2020).





Estimated source location of GW190814

Stochastic sampling

Draw samples from posterior and histogram them!

Efficient algorithms for sampling

- Markov-chain Monte Carlo (MCMC)
- Nested sampling

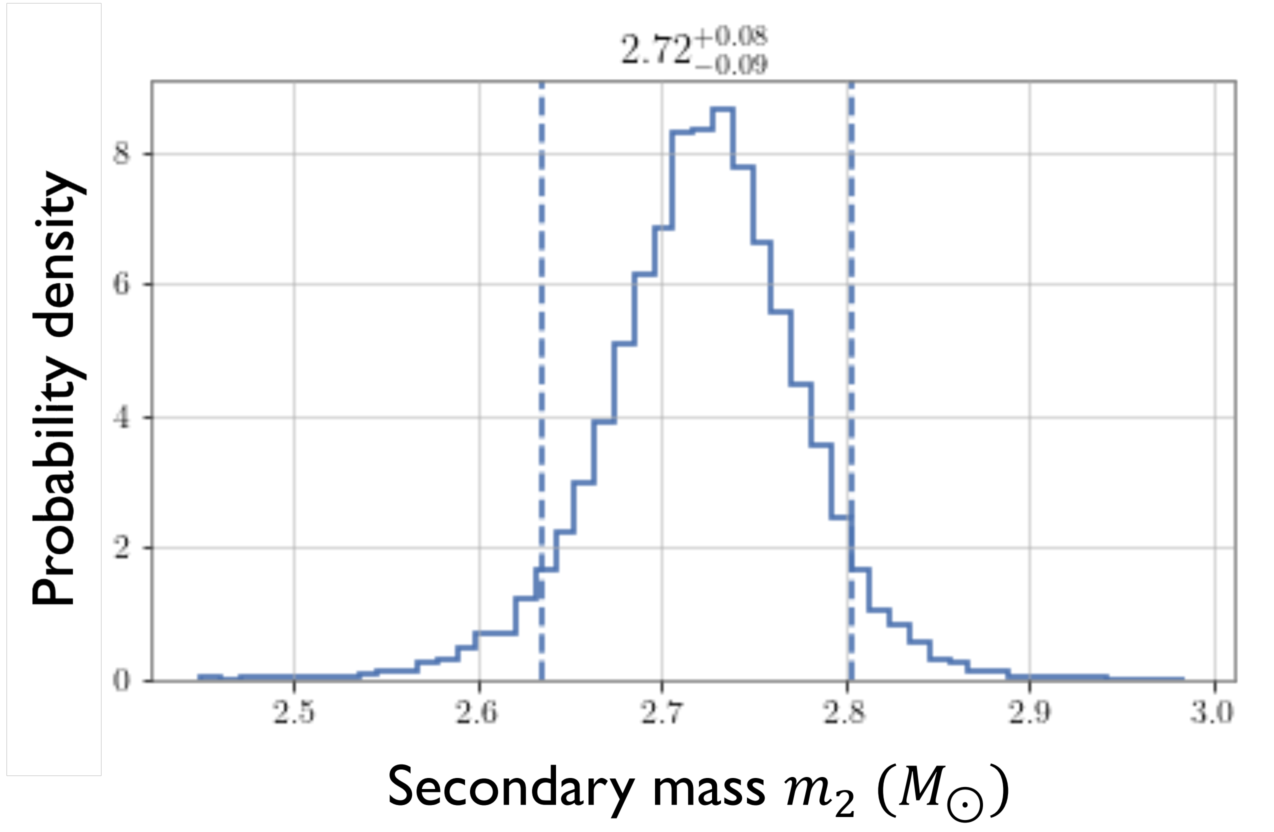
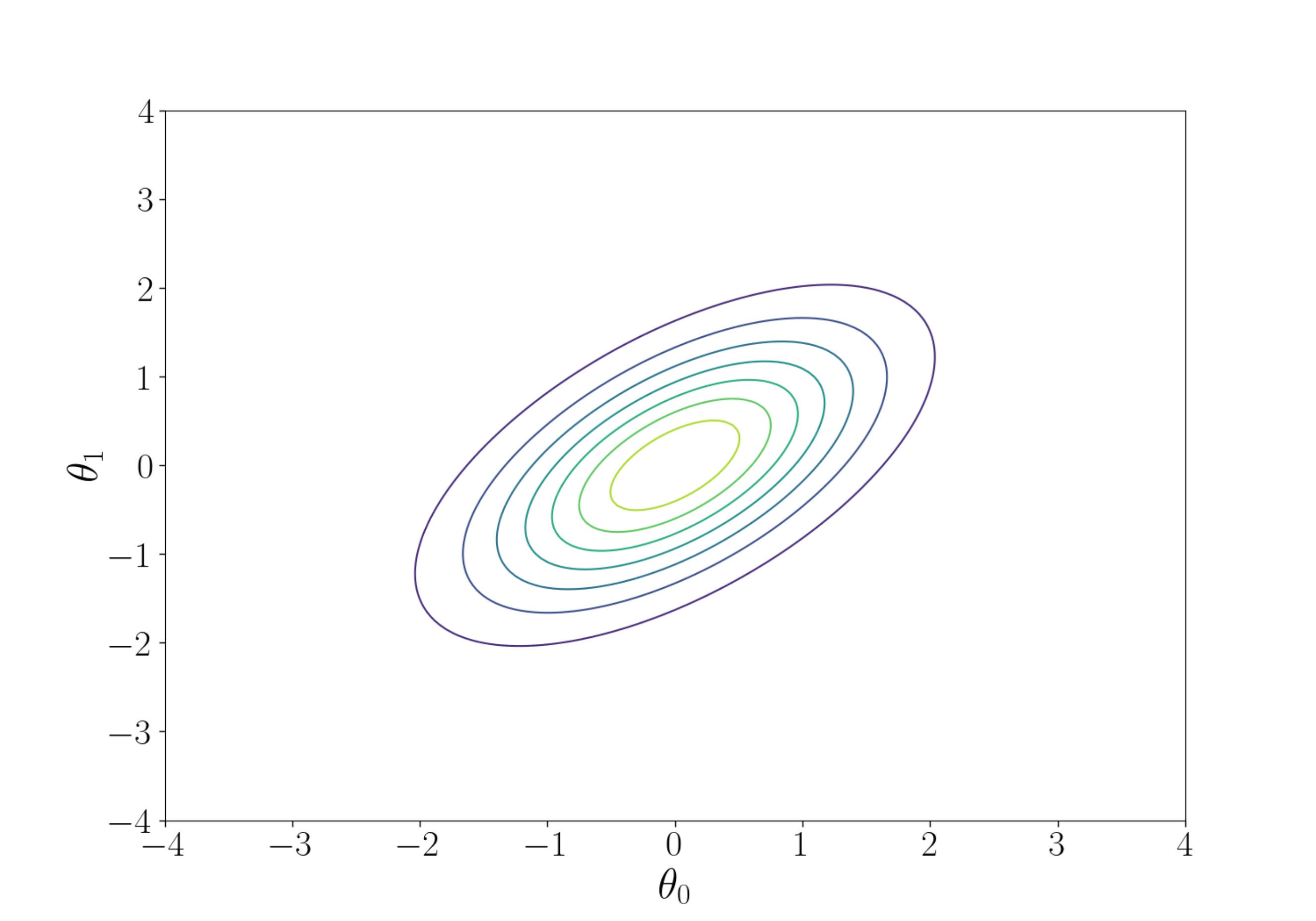


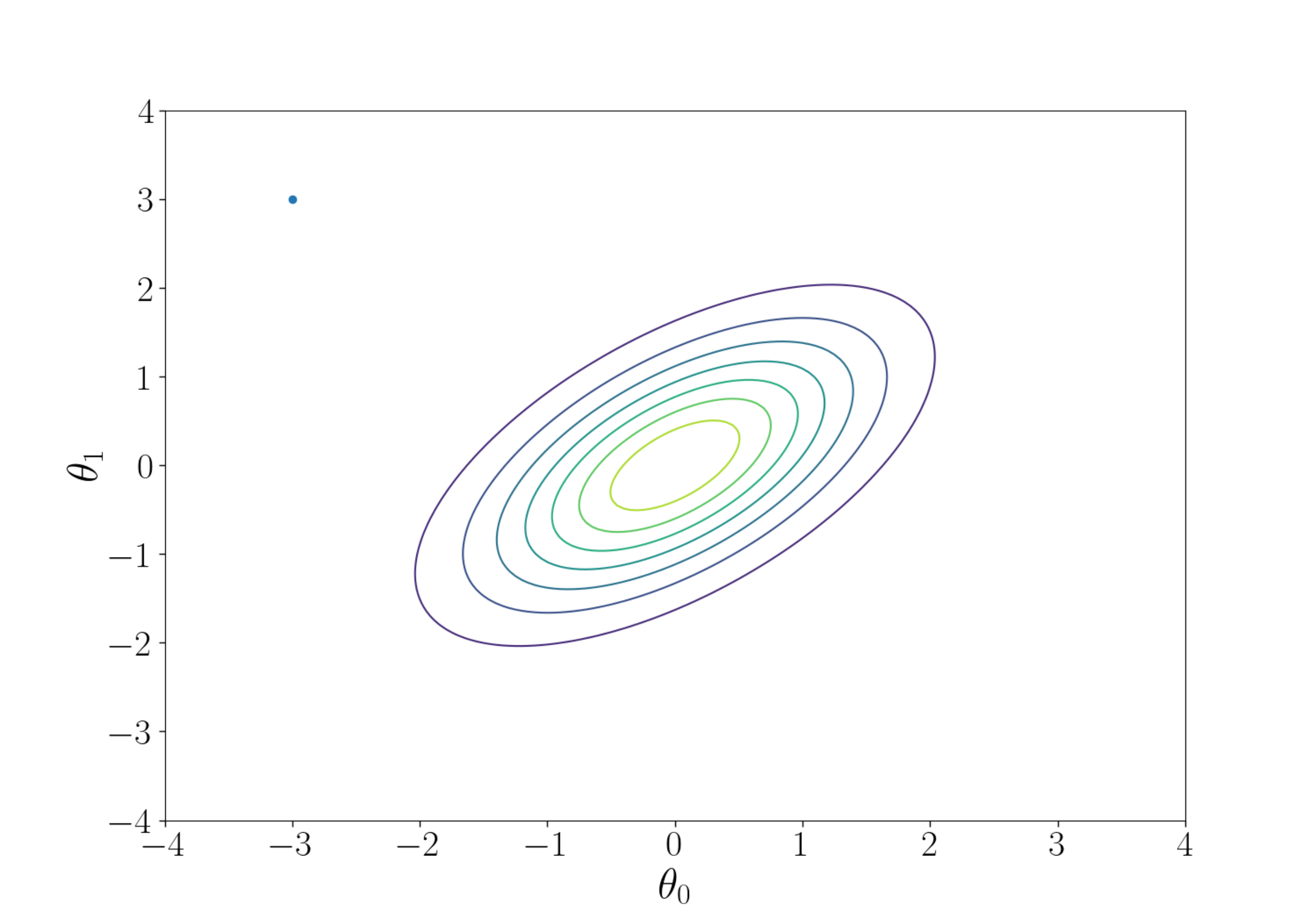
Figure: Estimated secondary mass of GW190814

MCMC example: Metropolis-Hastings algorithm



MCMC example: Metropolis-Hastings algorithm

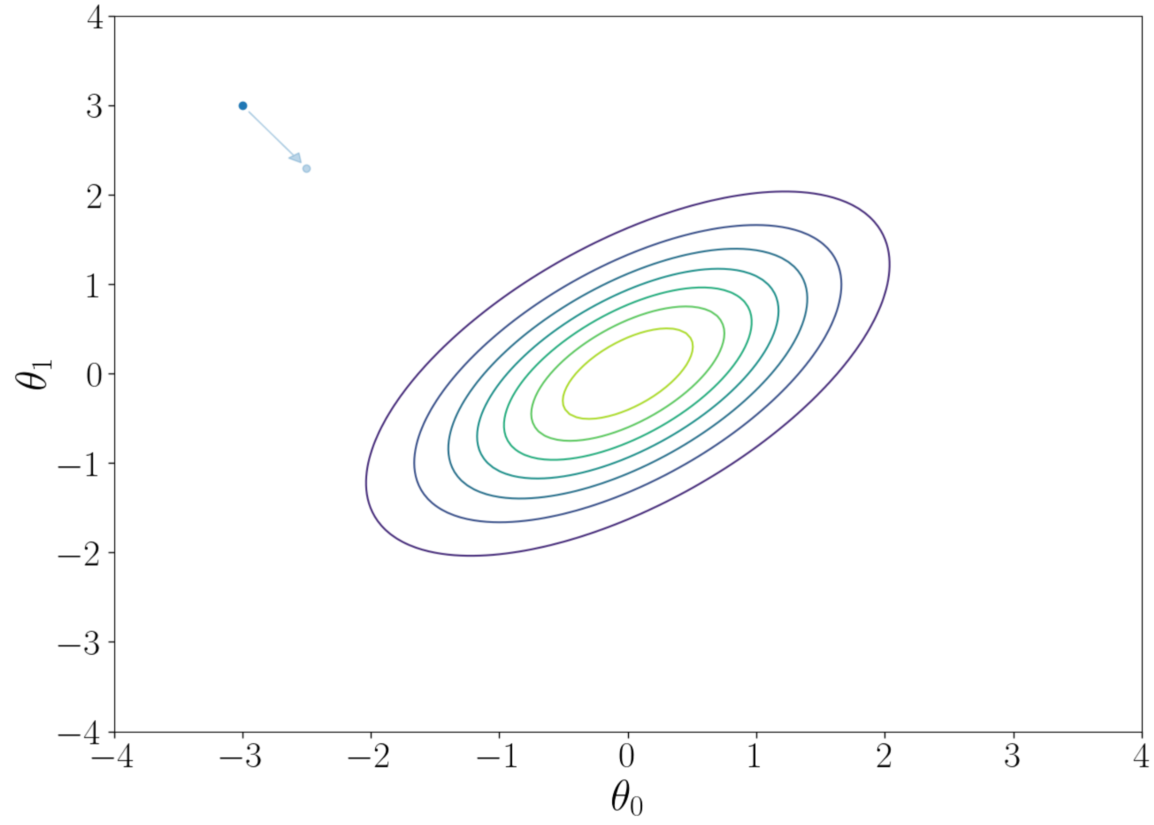
Start from a random point θ.



MCMC example: Metropolis-Hastings algorithm

Start from a random point θ.

Propose a next point θ' with proposal distribution $Q(\theta \rightarrow \theta')$.
Accept that proposal with probability of $\min\left\{1, \frac{p(\theta'|d,M)Q(\theta' \rightarrow \theta)}{p(\theta|d,M)Q(\theta \rightarrow \theta')}\right\}$.



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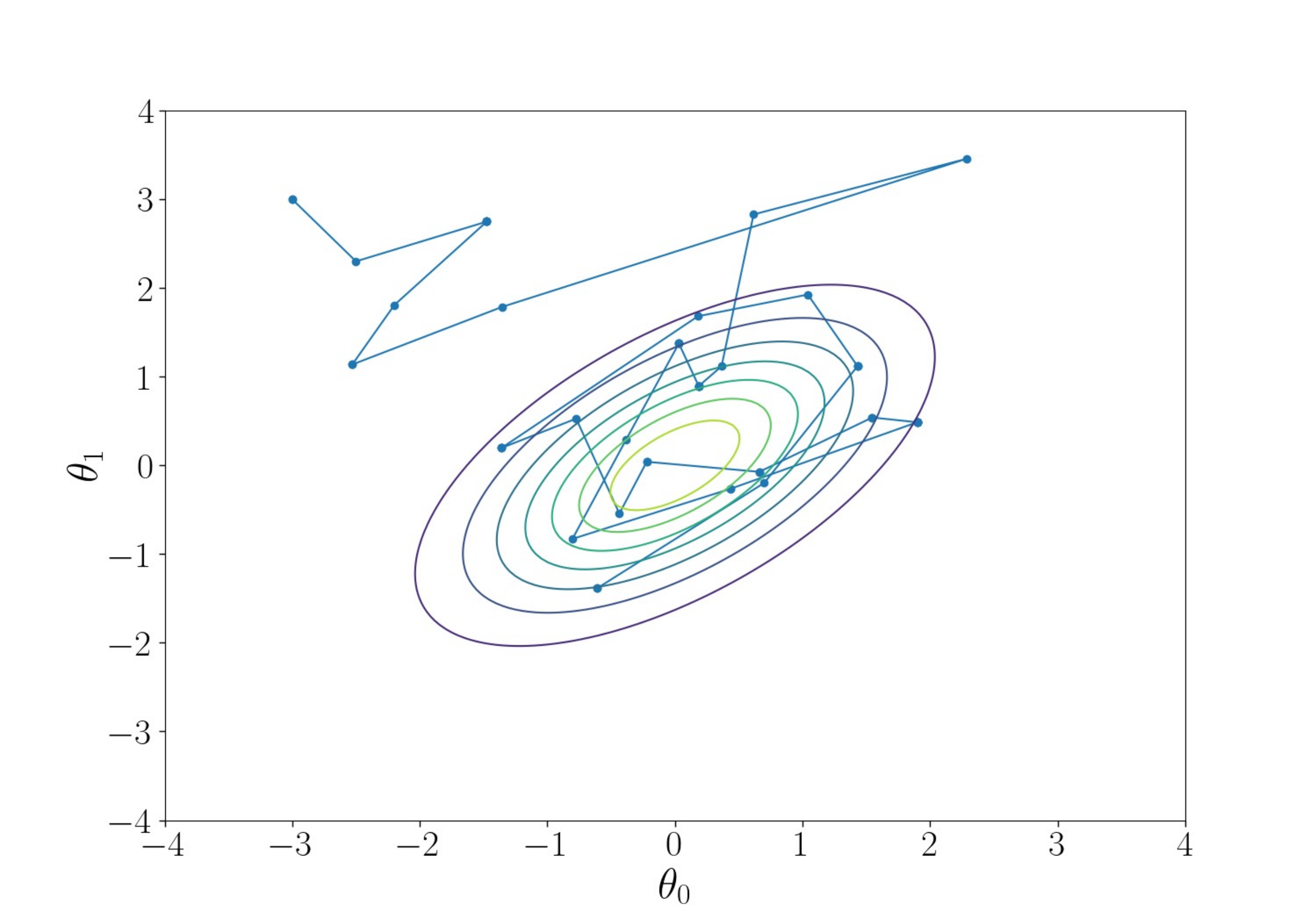
Repeat this proposal-acceptance.



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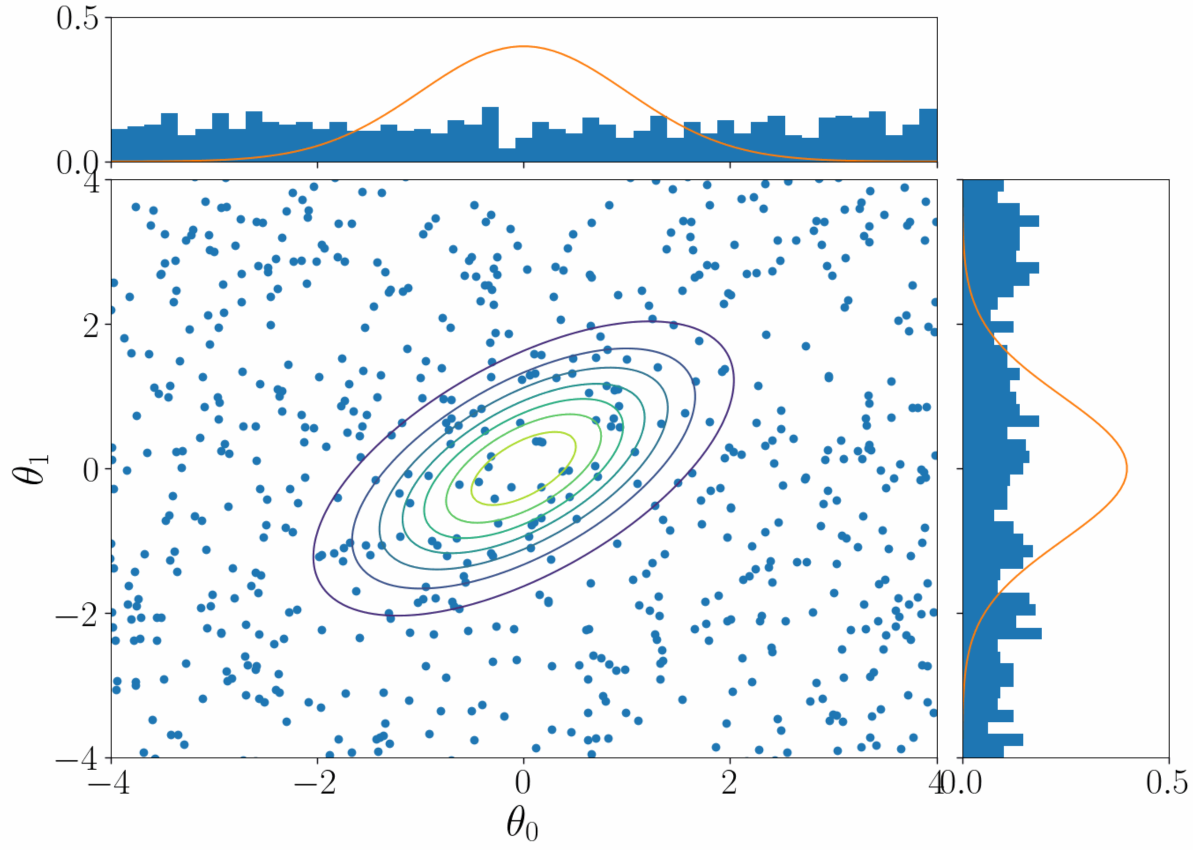
Repeat this proposal-acceptance.

The random point converges to a sample following posterior distribution.



Various open-source samplers

MCMC sampler

- emcee: https://emcee.readthedocs.io/
- ptemcee: https://github.com/willvousden/ptemcee
- PyMC: https://www.pymc.io/
- zeus: https://zeus-mcmc.readthedocs.io/
- ….

Nested samplers

- dynesty: https://dynesty.readthedocs.io/en/latest/
- nessai: https://github.com/mj-will/nessai
- Nestle: http://kylebarbary.com/nestle/
- pymultinest: https://johannesbuchner.github.io/PyMultiNest/index.html
- ….

Bilby: a user-friendly Bayesian inference library

References:
Ashton+ ApJS **241** 27 (2019),
Romero-Shaw+ MNRAS **499** 3 (2020).

- Python codes, installable with pip/conda.
- All the components necessary for CBC parameter inference built in (likelihood, frequently-used priors, useful parameter conversion functions etc.)
- Supports open-source samplers and the native one: bilby-mcmc.
- Can be used for non-CBC problems (See Tutorial 3.1).
- Can simulate CBC signals as well as analyzing real data (See Tutorial 3.2).

Playing with posterior samples

Posterior samples have been released from LVK.

- O1, O2: https://dcc.ligo.org/LIGO-P1800370/public
- O3a: https://zenodo.org/record/6513631
- O3b: https://zenodo.org/record/5546663

In [2]: samples

Out[2]:

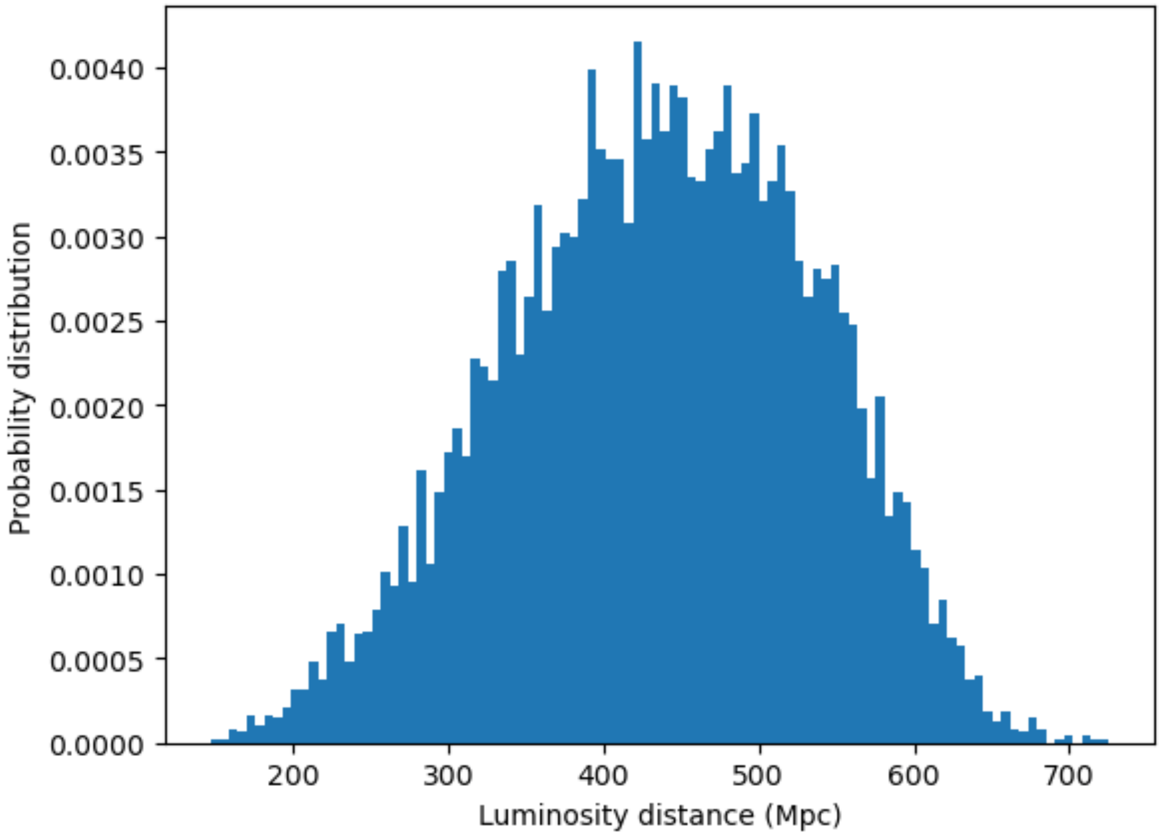
	costheta_jn	luminosity_distance_Mpc	right_ascension	declination	m1_detec
0	-0.976633	517.176717	1.456176	-1.257815	
1	-0.700404	401.626864	2.658802	-0.874661	
2	-0.840752	369.579071	1.106548	-1.136396	
3	-0.583657	386.935268	2.077180	-1.246351	
4	-0.928271	345.104345	0.993604	-1.069243	
...	...	...	...	...	
8345	-0.691637	306.985025	1.485646	-1.269228	
8346	-0.834615	462.649414	2.065362	-1.265618	
8347	-0.911463	448.930876	1.536913	-1.257956	
8348	-0.856914	561.020036	2.367289	-1.211824	
8349	-0.919556	519.641782	1.916675	-1.250801	

8350 rows × 10 columns

Playing with posterior samples

In [3]:
```
import matplotlib.pyplot as plt

plt.hist(samples["luminosity_distance_Mpc"], density=True, bins=100)
plt.xlabel("Luminosity distance (Mpc)")
plt.ylabel("Probability distribution")
plt.show()
```



Histogram of samples gives 1D posterior distribution.

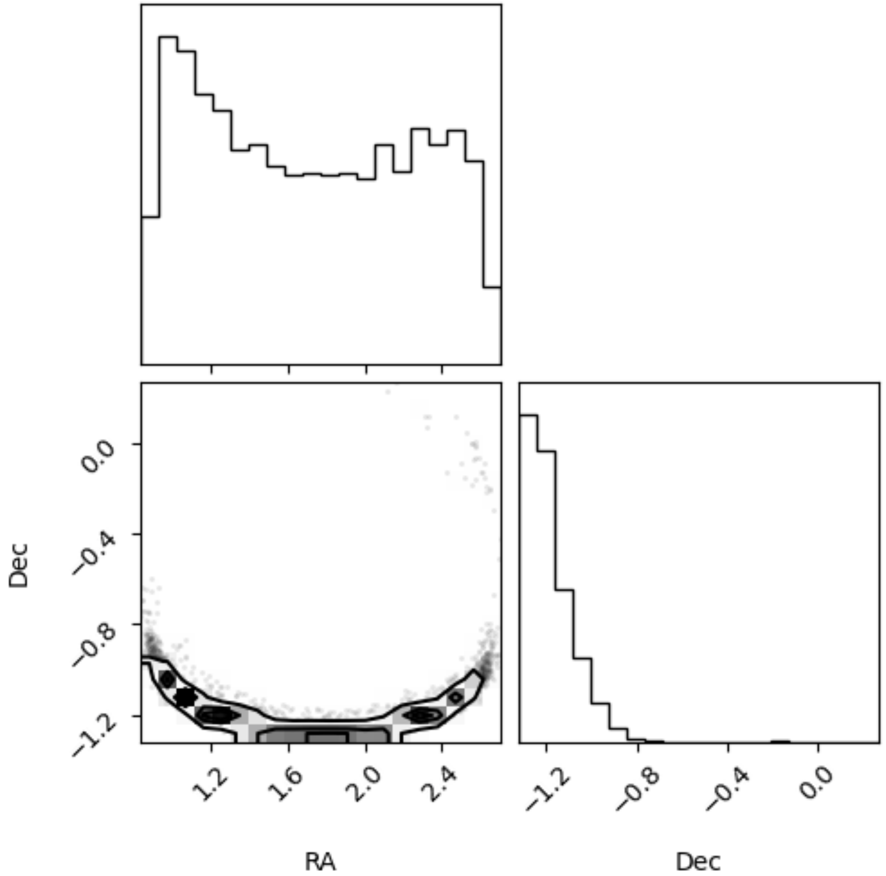
The 90% credible interval can be obtained by calculating the 5th and 95th percentiles of samples.

Playing with posterior samples

In [11]:
```python
import corner
import numpy as np

corner.corner(
    np.array([samples["right_ascension"], samples["declination"]]).T,
    labels=["RA", "Dec"]
)
```

Out[11]:



2D histogram is useful to understand correlation.

See Tutorial 3.3 to learn more about reading/plotting samples.

Conclusion

- Source parameters such as masses, spins, and tidal deformability parameters of colliding objects can be measured with observed gravitational-wave waveform.
- Source parameter estimation is typically performed with Bayesian inference, where likelihood is computed under the assumption of stationary Gaussian noise.
- We generate random samples following Bayesian posterior probability distribution and make their histograms to estimate source parameter values we are interested in.
- Useful references
  - Bilby documentation: https://lscsoft.docs.ligo.org/bilby/
  - Thrane and Talbot (2019): https://arxiv.org/abs/1809.02293

Tests of general relativity (GR)

Introduce parameters controlling deviation from GR predictions:

$$\tilde{h}(f) = A(f)e^{i\Phi(t)}, \qquad \Phi(t) = \Phi_{GR}(t) + \Delta\varphi_n f^{\frac{n-5}{3}}.$$

Credit: B. P. Abbott et al., PRL **123**, 011102 (2019).

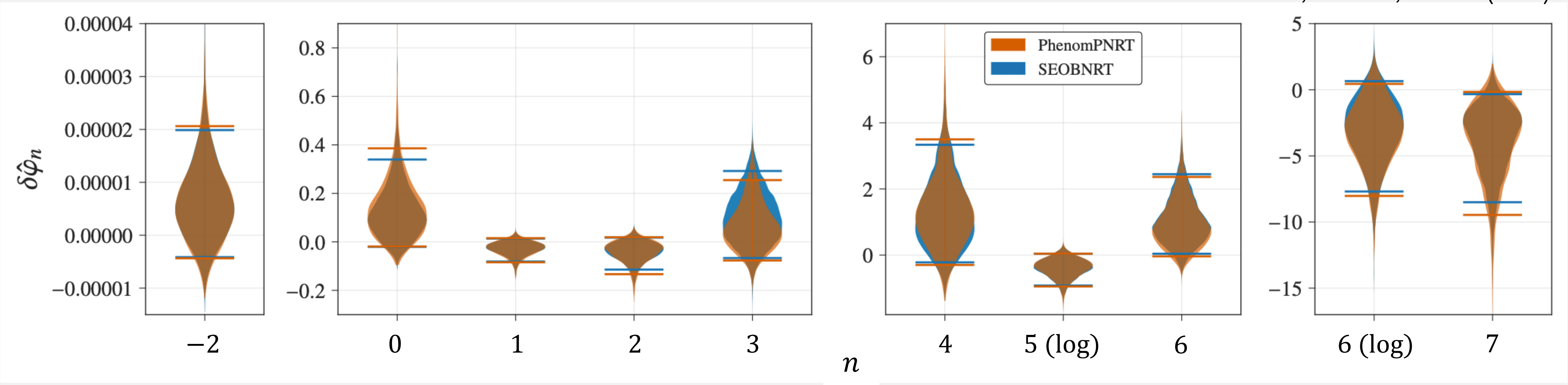


Figure: Constraints on deviation of GW170817 from GR predictions