Detecting Non-Power Law Stochastic Gravitational Wave Background

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The Stochastic Gravitational Wave Background (SGWB) is a consistent signal composed of a combination of many unknown sources. Since the SGWB is continuous, there is information on a much larger scale with the hope of included remnants of the early universe in the background. Current models work well to describe SGWB with current detector sensitivity where SGBW can be described by a simple power-law. However, common theories predict a turnover that will be detected with future detectors' sensitivity; this will lead to inconsistencies if current models are used. Since there is so much we do not know yet of the unknown sources it is pivotal to design a general and generic model to detect a SGWB that does not characterize as a simple power law. We use a new method of the Bayes factor along with `westley`, to do generic fitting when describing non-power law models, to detect SGWB. We will use splines and Gaussian processes to define this generic model and test with simulated data.

I. INTRODUCTION

Gravitational waves (GW) are ripples in space time that are initiated from extremely energetic sources. Known sources, in increasing order of how difficult they are to detect, include chirps from coalescing binary systems, periodic sources from pulsars, and bursts from supernovae [1]. Sources that are random, with multiple uncorrelated events, are called stochastic gravitational-wave backgrounds (SGWB). Unlike deterministic sources that last for a certain amount of time, the SGWB are always present. The SGWB can include events directly following the big bang, or more recently-generated signals that we can't necessarily individually detect.

LIGO's first detection in 2015 was a groundbreaking discovery that resulted in a Nobel prize. These individual events can inform us about the occurrences of stellar objects in the nearby Universe. However, looking at the SGWB can provide information on a much larger scale with much hope on the early universe. Models are created to predict how certain sources contribute to a SGWB and when an SGWB might be seen. We can do the reverse, and use the data to estimate the values of the parameters associated with each model.

Some models do not have a simple functional form such as a SGWB from a binary coalescences where the signal is generated from many individual events adding together. With current detectors the collection of binary coalescences can be well-described by the simple analytic power law [2]. However, as detectors become more sensitive, we expect to see a "smooth" turnover, which we can not be describe analytically. Additionally, we may see compact binary coalescences (CBCs) contribute a similar amount to the SGWB as other sources, like those from the Big Bang. Although each source might be described by a power law, its sum is not so easily analytically described. Consequently, we need a method that will characterize a SGWB of any "smooth" type.

II. BACKGROUND

A. Current Models

Currently it is assumed that the GWB spectrum is a power law,

$$\Omega_{GW}(f) = \Omega_{GW}(f_{ref}) \left(\frac{f}{f_{ref}}\right)^{\alpha}. \quad (1)$$

Where $\Omega_{GW}(f)$ is the GW energy density, f_{ref} is a reference frequency and α is the spectral index of the signal. Ω_{GW} and α are estimated. Although Ω_{GW} is usually considered a cosmological quantity here it is also used to describe the energy from astrophysical events so that we can compare them to cosmological sources [3].

Current methods to identify the GWB are signal to noise ratio (SNR) and the Bayes Factor. SNR uses the ratio of signal to noise where there is more weight on frequencies that have less uncertainty. This is done by inverse noise weighting, where C is the signal and σ is noise.

$$SNR(f) = \frac{\hat{C}(f)}{\sigma(f)},$$

$$C_{TOT} = \frac{\sum_{i=1}^{N} \hat{C}(f)/\sigma(f)^2}{\sum_{i=1}^{N} \sigma(f)^{-2}}; \quad (2)$$

$$\sigma_{TOT} = (\sum \sigma(f)^{-2})^{-\frac{1}{2}}$$
$$SNR_{TOT} = \frac{C_{TOT}}{\sigma_{TOT}}$$

The SNR tells us how greater the signal is to the detector noise. For GWB searches, when SNR= 3 it is considered as evidence of a GWB and a SNR=5 is a detection of a GWB. Bayes factor is similar to SNR in which the factor is that the data contains a signal divided by the probability the data is consistent with just noise in the detector.

$$Bayes\ factor = \frac{P(\hat{C}|\text{signal})}{P(\hat{C}|\text{noise})} \tag{3}$$

The probability is based on the posterior probability of the algorithm's parameters given the data. θ is used for signal and 0 is for noise,

$$\begin{aligned} P(\vec{\theta}|\hat{C}_i) &= \frac{P(\hat{C}|\vec{\theta})P(\vec{\theta})}{P(\hat{C})} \\ P(0|\hat{C}_i) &= \frac{P(\hat{C}|0)P(0)}{P(\hat{C})} \end{aligned} \tag{4}$$

The evidence is then give by the integral over the numerator. So the integral for signal in eq.4 is,

$$P(\hat{C}|\text{signal}) = \int d\vec{\theta} P(\hat{C}|\vec{\theta})p(\vec{\theta}). \tag{5}$$

In our case for a power-law of SGWB, shown in eq1 the parameters in $\vec{\theta}$ is amplitude A and spectral index α .

Both the SNR and Bayes factors perform well with current detectors and are the base work for a newly proposed method for future detectors. The interpolation model is the method where the height of the control points are interpolated between the parameters of $\vec{\theta}$, this will be further discussed in section III.

In figure 1 we show different versions of the SGWB from compact binary coalescences with the assumption of it made up completely of mergers, along with the estimated GWB sensitivity of LIGO detectors. The curves in Fig 1 are well-described power-laws until a certain point were they turn over. The gray sensitivity curves show that for realistic curves (i.e. those with chirp masses below 50 $M_{\odot}$), current generation detectors are not sensitive to this turnover.

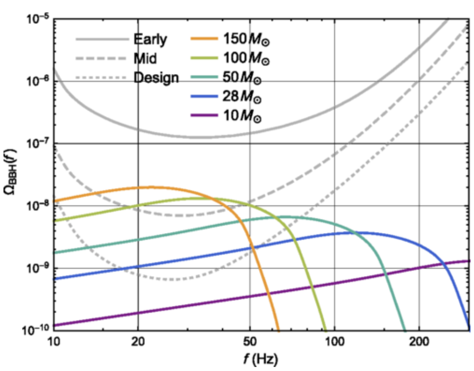

Figure 1. Figure reproduced from [2]. We show the binary black hole's background with various chirp masses with the Fiducial model for SGWB (colored lines). Power-law integrated curves for one year with Advanced LIGO (grey lines).

This turnover is the astrophysical GWB's non-analytical piece. The turnover depends on the masses of the black holes and becomes more complex as different populations are added to it. Future detectors will be sensitive to these turnovers which can clarify characteristics for CBCs such as: the time it takes for a star to merge in a binary, if properties of the universe contribute to the formation and/or mass of a black hole, and how the populations of masses and spins of a neutron star and a black hole that enter a binary look like in our spectrum. In addition, there is an unknown cosmological background in CBCs that are not black holes, which motivates another reason to create a new model since we simply do not know what the sources are and how to define them.[3].

B. Evolution of Models as Detectors Improve

Once detectors become more sensitive there will be an abundance of individual events. To search for the SGWB, we will subtract the loudest events from the data, which will then change the spectrum of the SGWB. An example of an expected SGWB after subtracting individual events is shown in Fig 2. Evidently the spectrum is no longer a straight line in log space but it does appear "smooth".

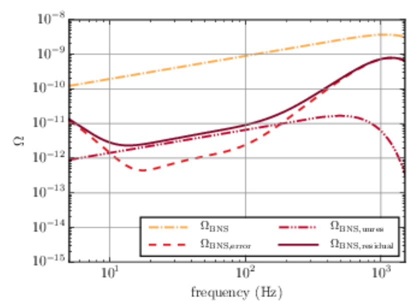

Figure 2. Figure reproduced from [4], the SGWB from all neutron stars (doted orange line) plotted together with background from unresolved (unsubtracted) neutron stars (dotted red line), and the sum of the two (red sold line). With the neutron star summed with unresolved background the line is now longer a power law.

A general and generic model can effectively detect a SGWB of any "smooth" shape at higher sensitivity. The paper [4] predicts most models with a "smooth" look therefore, a new model must capture similar values or change smoothly from one frequency bin to the next. We propose two methods to detect generic models. Both methods are regularly used to fit smooth looking curves, spline fitting and Gaussian process. As we will discuss in the next section, this new model can also be used to develop a consistency check to verify a detection.

III. METHODS

To detect a SGWB we will cross-correlate data between detectors. In the following equations the tilde indicates the use of the Fourier transformation. The detector data ($\tilde{s}_i$) involves both GW signal ($\tilde{h}$) and noise ($\tilde{n}$) with the dependency of frequency,

$$\tilde{s}_1(f) = \tilde{h}_1(f) + \tilde{n}(f). \tag{6}$$

The data of the two detectors is then defined in a cross correlation statistic ($\tilde{C}(f)$) in every frequency bin [5],

$$\tilde{C}(f) = \frac{2}{\tau}\frac{Re[\tilde{s}_1^*(f)\tilde{s}_2](f)}{\gamma_f S_0(f)}, \tag{7}$$

where Re indicates the real part of the cross correlation, τ is the time over which we are analyzing data, and the normalization includes cosmological constants, as well as the overlap reduction function $\gamma(f)$, which we discuss soon. We can substitute Eq 7 into Eq 6 and take an average:

$$\langle\tilde{s}_1^*(f)\tilde{s}_2(f)\rangle = \langle\tilde{h}_1^*(f)\tilde{h}_2(f)\rangle + \langle\tilde{n}_1^*(f)\tilde{h}_2(f)\rangle + \langle\tilde{h}_1^*(f)\tilde{n}_2(f)\rangle + \langle\tilde{n}_1^*(f)\tilde{n}_2(f)\rangle.$$

We then assume that the signal is uncorrelated with detector noise and the noise between the two detectors is uncorrelated. Therefore,

$$\langle\tilde{s}_1^*(f)\tilde{s}_2(f)\rangle = \langle\tilde{h}_1^*(f)\tilde{h}_2(f)\rangle. \tag{8}$$

Next, we note

$$\frac{2}{\tau}\langle\tilde{h}_1^*(f)\tilde{h}_2(f)\rangle = H(f)\gamma(f). \tag{9}$$

Here $H(f)$ is called the gravitational wave power, and the proportionality constant $\gamma(f)$ is called the overlap reduction function. The overlap reduction function is a weight function in frequency that quantifies what fraction of the GW power our detectors are sensitive to. Thus, $\gamma(f) = 1$ means we see all of the GW power in our detectors, but $\gamma(f) = 0.5$ means we see only half of the GW power. Since we know exactly what detectors we are using its value is exactly known [3]. This frequency dependence provides a great insight into the frequencies to which the detectors are most sensitive.
When we substitute Eq. 9 back into Eq. 7, we find that in general,

$$\langle\tilde{C}(f)\rangle = \frac{H(f)}{S_0(f)} = \Omega_{GW}(f) \tag{10}$$

Where the constants $S_0(f)$ are used so that we have the cross-correlation proportional to the energy density.

The shape of the cross-correlation is what we want to model. $\tilde{C}(f)$ is calculated by taking the cross-correlation between our detectors for numerous short time intervals then taking the average of all the runs. We then compare the averaged $\tilde{C}(f)$ to power law spectra to verify if there is any evidence of power law. What we propose to do here, is to instead compare to more generic “smooth” functions like splines or Gaussian processes. Then proceed to test our model with simulated data.

A. Needed Tools

The LIGO collaboration has several libraries and pipelines that are used specifically for frequency domain spectra. Our work mainly uses the library `bibly` and `pygwb` as well as the pipeline `pygwb_pipe`. Our method uses Bayesian statistics, like `bibly`, and a hybrid approach, executed by `pygwb`. This library has several functions that are needed such as initializing a Power Spectral Density and generating a prior in log uniform and Gaussian distribution. `pygwb_pipe` is the main pipeline to work with SGWB that uses the library `pygwb`. We used this pipeline to simulate data from all three detectors Hanford, Livingston, and VIRGO to then inject a common GW signal with a chosen spectrum into our current data.

Another crucial code is the package `westley`. This code holds a big chunk of what we will be testing. Our model will have control points, referred to as knots, to interpolate between to create a model that we will fit to the data. With this package we can interpolate between the knots using either a piece-wise power law or a cubic spline and run it a number of times to find the probability distribution on where the knots should be placed and turned on to have the best fit. Rather than taking random guesses to find the probability distribution we use the technique Markov chain Monte Carlo (MCMC).

B. Statistics

An MCMC is used to obtain a preferred probability distribution on some set of unknown parameters. For our model, we want to find the probability distribution on the height of each knot and how many are used. With MCMC we are able to make intelligent guesses to where the knots should be placed. A new knot's position is accepted depending on how well the ”new” guess will let the model fit the data compared to its current, ”old” guess. The likelihood function L tells us how well the model fits the data, and we use a Gaussian likelihood function. The placement of the knot is kept whether,

$$P(accept) = 1, \text{if point has } L_{new} > L_{old}$$
$$P(accept) = \frac{L_{new}}{L_{old}}, \text{if point has } L_{new} \leq L_{old}$$

The proposed distribution function tells the algorithm how we decided to guess. The way we guess does not interfere with the results, however, it does affect the efficiency of the algorithm time wise.

IV. PROPOSED BAYES FACTOR METHOD

This Method takes concepts from "SNR" and "Bayesian Factor" methods of searching for SGWB. Since the knots in our model are used to fit a signal, they should only be present when a signal is present. Therefore, we should have 1 or more knots when there is a signal present and no knots where there is only noise present. Our method is written as,

$$BF = \frac{N_{\geq 1}}{N_0}, \tag{11}$$

where $N_{\geq 1}$ is the number of samples with signal and N_0 is samples classified as only noise.

V. RESULTS

A. Testing Algorithm

To see if our algorithm works we tested it on an injected GWB. Our simulated data was reduced to just the signal ($\hat{C}$) and cross correlated with the Hanford and Livingston detector. We set the model to have a max of 20 knots and ran the MCMC for 100000 steps. From the 100000 steps, we discarded the first 50,000 as "burn in" and then recorded the parameters at only every 10^{th} step. Figure V A below shows a comparison between the injected signal (orange), the data (blue and green), and the recovered signal for 500 MCMC steps (black).

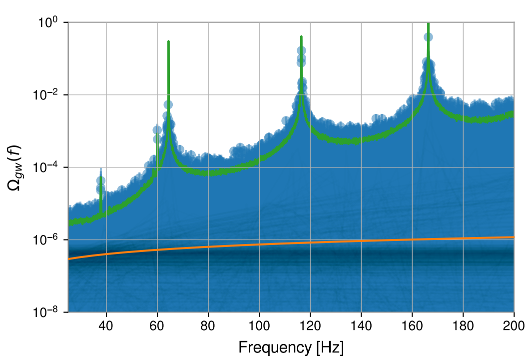

Figure 3. The figure shows the energy density estimated from the data in blue, the uncertainty in green, the energy density of our proposed injected power law in orange and the interloped knots in black. The MCMC with 500 steps, relatively fit the injected signal throughout the runs.

The plot tells us the energy density Ω_{GW} estimated from the data (blue) and its uncertainty (green). The energy density of the injected signal is shown in Orange. Since this is in log space our injection is mostly linear and most of the simulated runs from `westley` fit along the power law. Therefore the code has identified a GW. Figure 4 is a visual representation of which knots where in used in each run of the model.

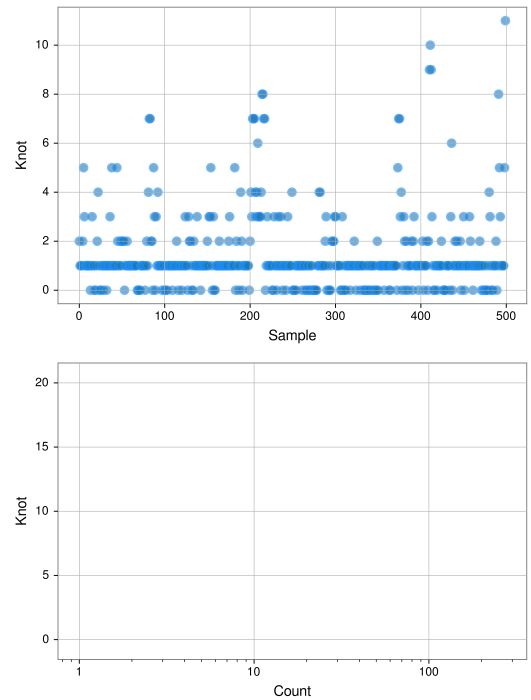

Figure 4. Top, shows when the knot was turned on during a certain run. Bottom, is a histogram to visualize how many times the certain knot was used for all runs.

The model we tested had spaced out the knots evenly throughout the spectra and made its own choice when the knots should be turned on or off. Figure 4 illustrates the height of each bin which we can use the sum of the heights for all the bins with the y-axis ≥ 1 and divide by the height of the bin at zero to get a log Bayes factor of around 1.3.

VI. NEW PROGRESS

After familiarizing with `westley` and confirming it's competence, the following steps were to start fitting different simple power-laws and broken power-laws. I created a script that fast tracked our analysis of how `westley`'s generic fitting works on different amplitudes Ω_{GW} for both simple and broken power-laws. Each trial had a sample of 100 with amplitudes covering the range of 10^{-7} to $4x10^{-7}$ and each trail went through 10^6 MCMC iterations. In the following sections I explain the scripts being used and how each model, simple power-law and non-power law, behaved as predicted with the generic fitting.

A. Code

The generic fitting built upon the hybrid analysis of `pygwb` takes around 20 minutes with only one sample at an MCMC iteration of 10^5 and the time increases as the samples and iterations increase. I began the past weeks analysis with a `for loop` to run several samples to see how different Ω_{GW} power-laws behaves with our outputs: SNR_{TOT} (eq.(2)), and $ln(BF)$(eq.(3) and (11)), and the error on that Bayes Factor.

This turned to be an inefficient route as the job's run time went as long as a full work day. To improve our analysis' efficiency I learned how to make as script to run with LIGO's cluster in the background while my computer was in sleep mode. This sped up our project timeline and allowed for a greater sample size and a more detailed MCMC.

B. Simple Power law

For simple power laws we expect that Ω_{GW} and detection have an equivalence correlation. The log Bayes factor is our reference of how strong the detection is since we use the number of samples N that have 1 or more knots turned on over the number of samples with no knots turned on eq.11. In figure 5, we see that as the amplitude increases so does the log Bayes factor.

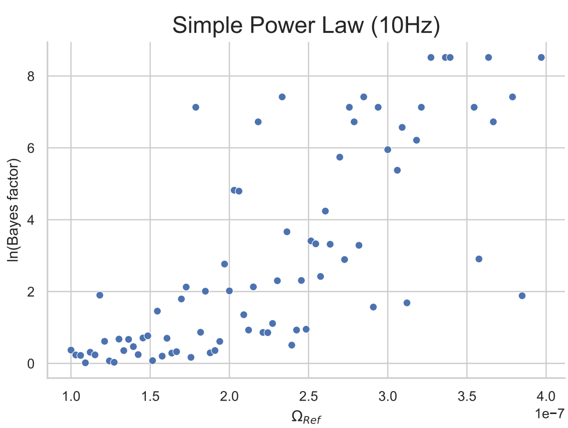

Figure 5. Here we have a simple power law of different amplitudes compared to the $ln(BF)$ provided by `westley`. As amplitude increases the corresponding Bayes factor reciprocates. As a product of this relationship, it is confirmed that our genetic fitting is working correctly.

Besides a few outliers, the general trend validates our prediction confirming the code is working. Once we had the right trend I tested our Bayes factor method to the SNR method (2) that is currently used with SGWB models, illustrated in figure 6.

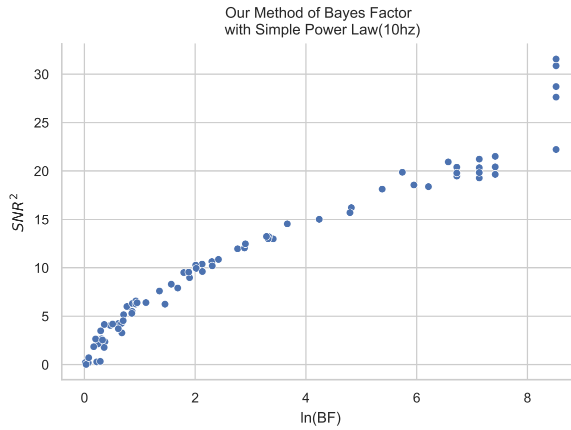

Figure 6. The figure shows our results from our 100 sample `westley` run. It is known that ln(BF) is proportional to SNR^2. Focusing on $SNR^2 \geq 10$ and $ln(BF) \geq 2$, our data depicts the linear relationship of SNR^2_{TOT} to ln(BF).

From the SNR and Bayes factor results we expected there to be a linear relationship between $ln(BF)$ and SNR^2_{TOT}. The figure above indicates the expected relationship within our sample, this verified the proficiency of our method. After `westley` worked adequately with a simple power law we then moved on to a broken power law.

C. Non-Power law

A broken power law is defined by a peicewise function were the different power-laws' boundaries are the "breaks" in the function. To switch the simple power law model to a broken power law model I changed the f_{knee} parameter of the code from 10Hz to 50Hz based on my group partner's project. At 50Hz there is a barely any detection of gravitational waves therefore, this test was to indicate if `westley` could or could not distinguish between simple power laws and broken power laws.

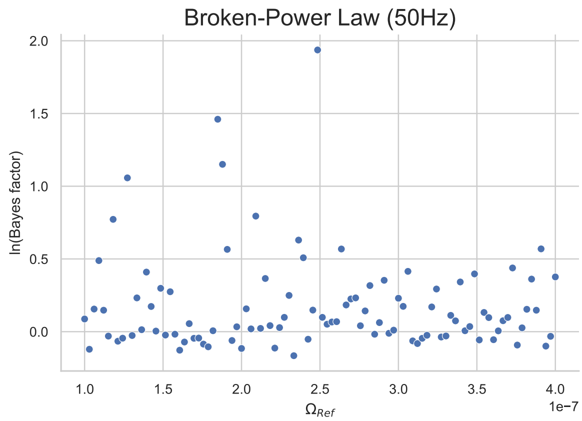

Figure 7. We show the ln(BF) comparing a broken power law model to just a power law mode. The majority of ln(BF) is localized at 0 resulting in no distinction between simple and broken power-law from `westley`.

From figure 7, despite the change in Ω_{GW} the log Bayes factor is localized at 0 for the majority of the samples. From this we concluded that `westley` can not differentiate a simple power law to a broken power law. Because of this we know that our fitter is working accurately and can move onto the next part of the project.

VII. NEXT STEPS

From `westley`'s performance of the passed trials the next step is conduct larger scaled runs at a faster pace using Condor. I will add more iterations to the model and run the samples in parallel to cut our job time further. These runs will provided a better confirmatory analysis of `westley`'s performance. We will also conduct trails to prove that we can detect non-power law SGWs and then merge my project with Taylor Knapp's project. Together our projects will answer what type of shape can the SGWB have and at what point does SGWB have ”the shape” compared to a simple power law?

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