Fisher Information Analysis for Emissivity Estimation

Hiya Gada
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Mentors: Radhika Bhatt, Rana Adhikari, Christopher Wipf

Flow of the Presentation

- Motivation
- Our system - The Cryostat
- Optimal System Identification
- Fisher Information Matrix
- Optimal Control Input
- Power Spectrum Optimization
- Optimal Experimental Configuration
- Future work
- Acknowledgement and References

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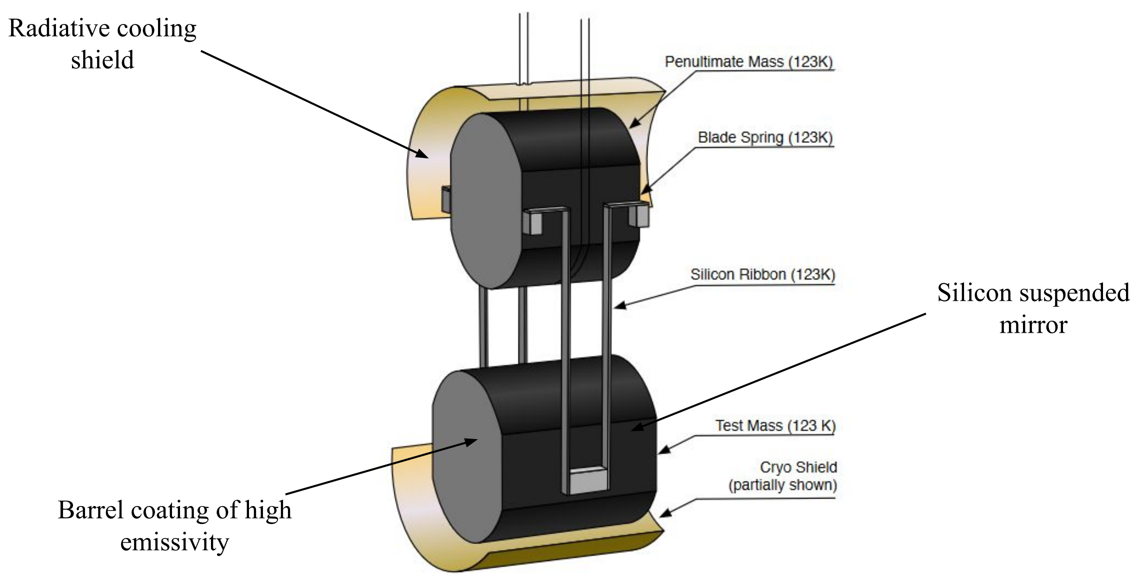
Motivation

Objective	Mariner (Voyager prototype) will have reduced thermal noise
	High emissivity coating will to increase the radiative coupling to its cold environment and dissipate laser power
	Estimating coating emissivity with least uncertainty for various materials



Motivation

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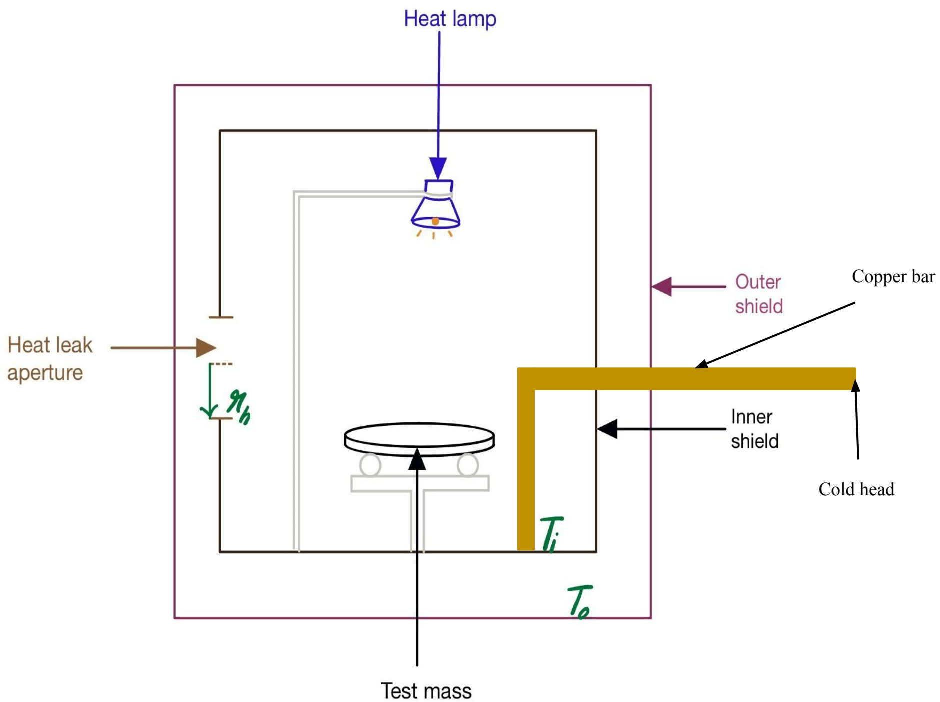
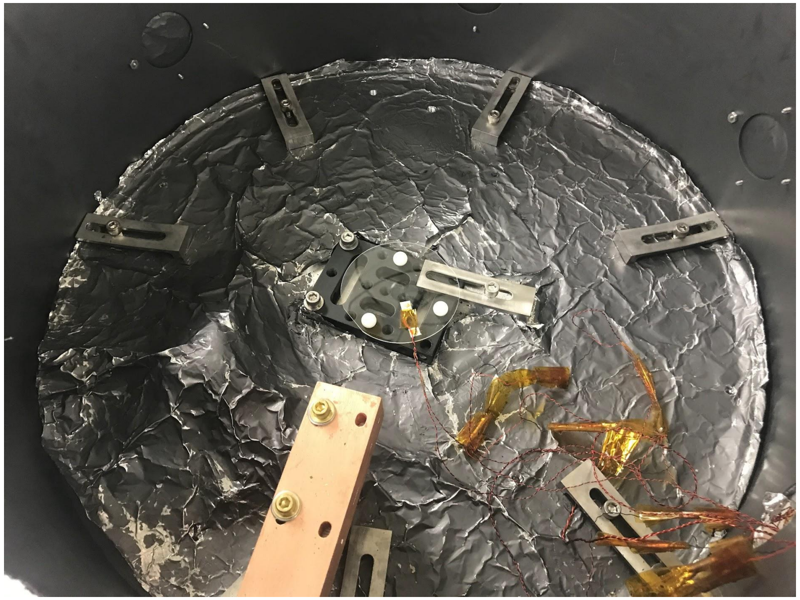
Problems	Time-constant for cool down is over a day
	Experiment can't be run multiple times to get expected value

Solution	Find and implement optimal experimental design and input to get most certain emissivity measurement

The Cryostat





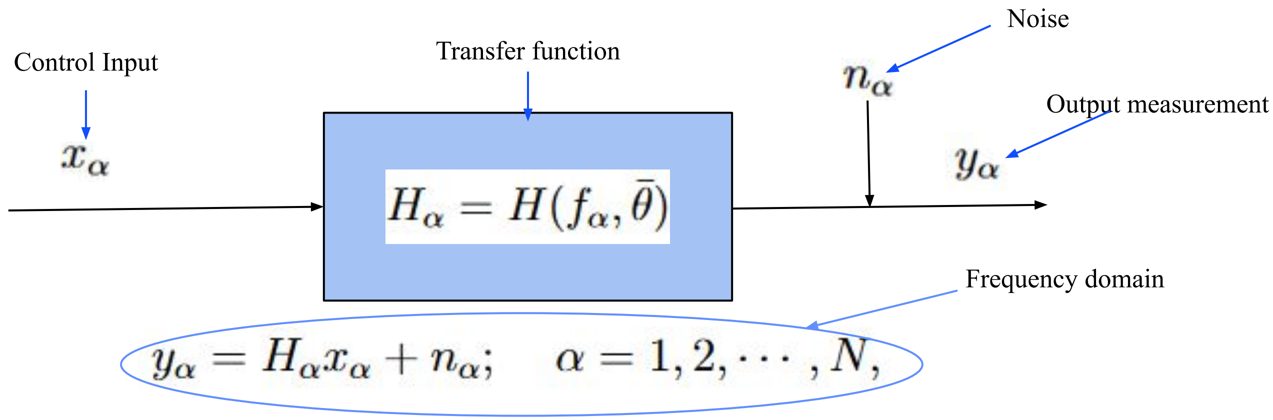
Cooling term due to inner shield

Heating term due to heat leak

Control heat input

$$mC_p\frac{dT_1}{dt} = \sigma A_1\left[\frac{T_2^4 - T_1^4}{\frac{1}{\epsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\epsilon_2} - 1\right)} + \frac{T_h^4 - T_1^4}{\frac{1}{\epsilon_1} + \frac{A_1}{A_h}\left(\frac{1}{\epsilon_h} - 1\right) + \frac{1}{F_{1\to h}} - 1}\right] + P(t),$$

Optimal System Identification



Main Question:

How should our input signal look like to determine the transfer function poles and zeros with least uncertainty and robust to noise?

- Frequency domain makes analysis simpler for linear systems
- Back calculate parameters from poles and zeros
- Hence, identify system

Intuition

Given model: $y = mx + c$

Only two measurements allowed

What x values should we measure at to estimate m and c?

When x values are farthest apart!

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Fisher Information Matrix

The output measurement:

$$y_\alpha = H_\alpha x_\alpha + n_\alpha; \quad \alpha = 1, 2, \cdots, N,$$

Corresponding to frequency

Transfer function

Noise

Fisher Matrix formulation:

$$\mathcal{F}_{ij} = \sum_\alpha \frac{1}{\sigma^2_{H_\alpha}} \mathrm{Re}\left[\frac{\partial \hat{H}^*_\alpha}{\partial \theta_i}\frac{\partial \hat{H}_\alpha}{\partial \theta_j}\right]\Bigg|_{\bar{\theta}},$$

$$\sigma_{H_\alpha} = \frac{|n_\alpha|}{|x_\alpha|}$$

Parameter prior

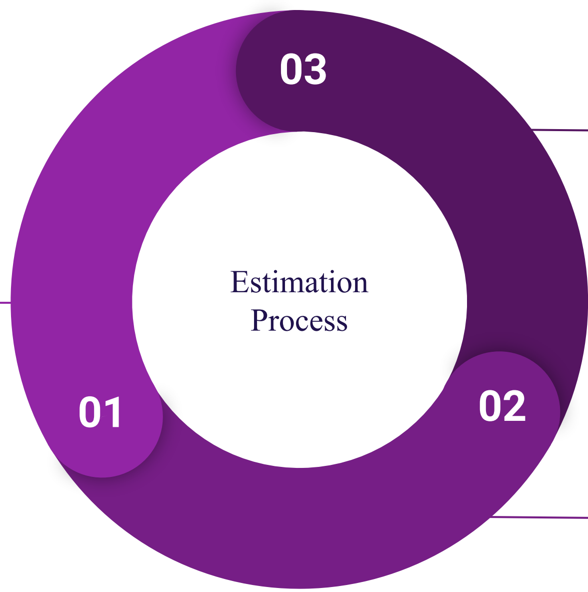
System parameters

Cramer-Rao bound:

$$\mathcal{C} \geq \mathcal{F}^{-1},$$

Covariance matrix

Maximize Fisher Information (minimize covariance matrix) with parameter priors



Run experiment with the optimal input and parameters to take the measurement

Obtain optimal input and parameters estimates

Optimal Control Input

The linearized system:

$$mC_p\frac{dT_1}{dt} = \sigma A_1\left[\frac{T_2^4 - T_1^4}{\frac{1}{\epsilon_1} + \frac{A_1}{A_2}\left(\frac{1}{\epsilon_2} - 1\right)} + \frac{T_h^4 - T_1^4}{\frac{1}{\epsilon_1} + \frac{A_1}{A_h}\left(\frac{1}{\epsilon_h} - 1\right) + \frac{1}{F_{1\to h}} - 1}\right] + P(t),$$

$$\dot{T}_1(t) = AT_1(t) + P(t)$$

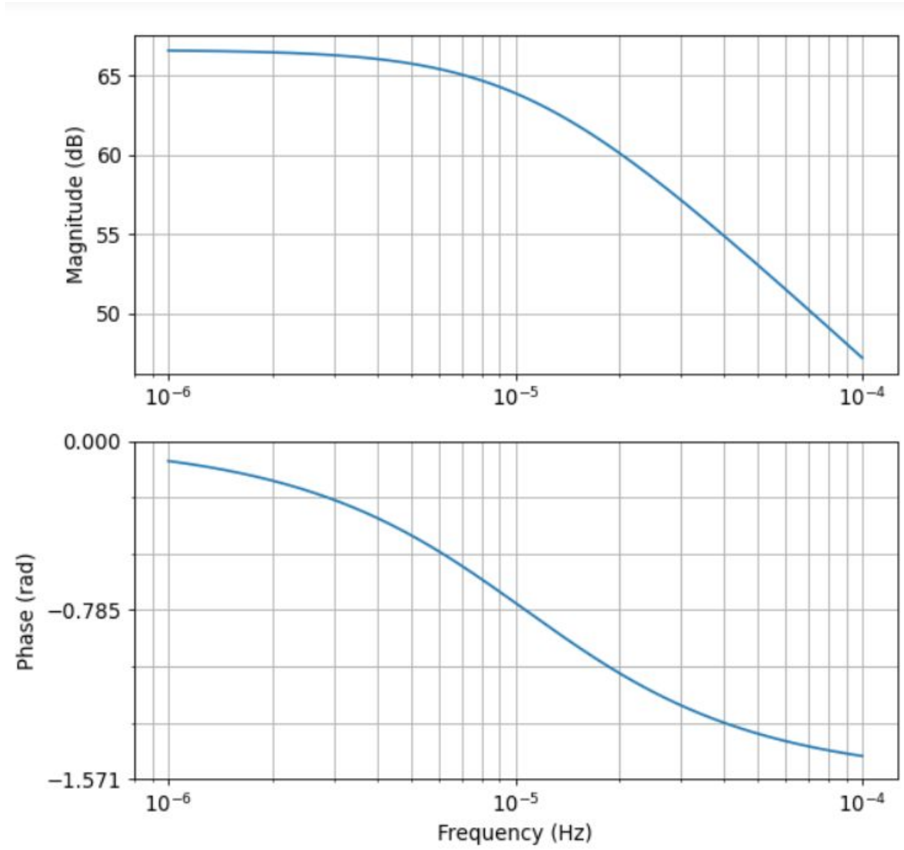
The transfer function:

$$H_T(f_\alpha) = \mathcal{L}\left(\frac{T_1(t)}{P(t)}\right)$$

Things to note:

- The cut-off frequency is in 10^{-5} Hz
- This corresponds to ~27 hours
- Any frequency above 10^{-5} Hz will be damped
- We must analyse around this region

Bode Plot



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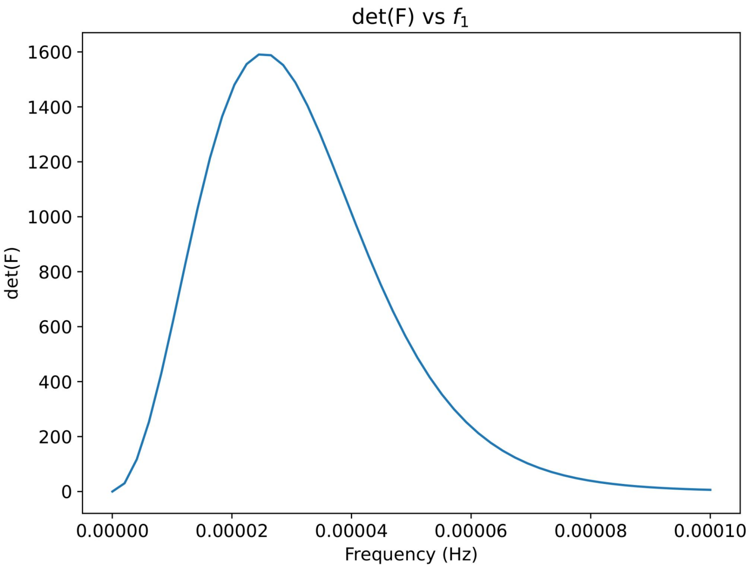
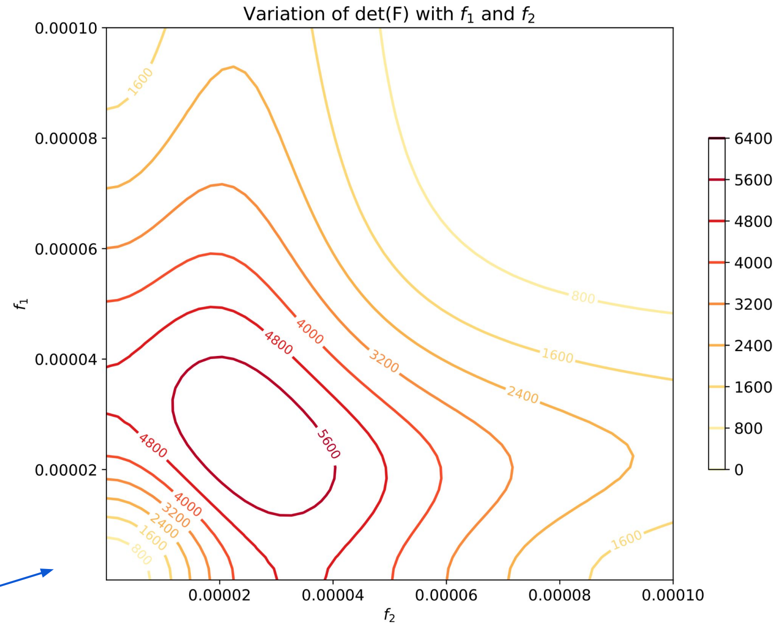
Optimal Control Input

For a one frequency input signal:



For a two frequency input signal:



Optimal frequency is ~ 10^{-5} Hz

- Assigning same amplitude and noise to both frequencies gives us degenerate optimal frequencies.
- Considering frequency-dependant noise and not white noise will give us non-degenerate frequencies
- Assigning different amplitudes or power to the two frequencies will also give non-degenerate frequencies

Power Spectrum Optimization

Some constant for total power

Input power spectrum: $\chi(\Omega) = (|U(1)|^2, \cdots, |U(F)|^2)$ with $\sum_{k=1}^{F} |U(k)|^2 = \mathfrak{P}$

Where frequencies are: Ω_k for $k = 1, \cdots, F$

Dispersion function:

$$\nu(\chi, \Omega_k) = \text{trace}([\mathcal{F}(\chi)]^{-1} fi(\Omega_k)),$$

Fisher Matrix

Power concentrated at one frequency

Algorithm:

Distribute total input power equally over a set of discrete frequencies in our band of interest: $\mathbb{F} = \{\Omega_1, \cdots, \Omega_F\}$

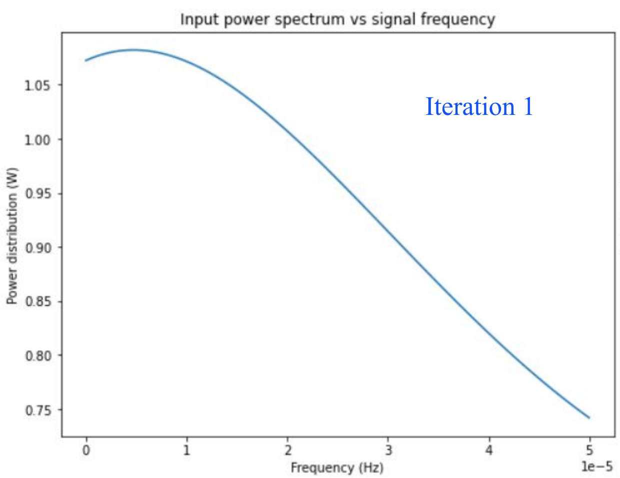
Set $i = i + 1$ in the algorithm and for the current IPS, calculate the dispersion function: $\nu(\chi_i, \Omega_k)$

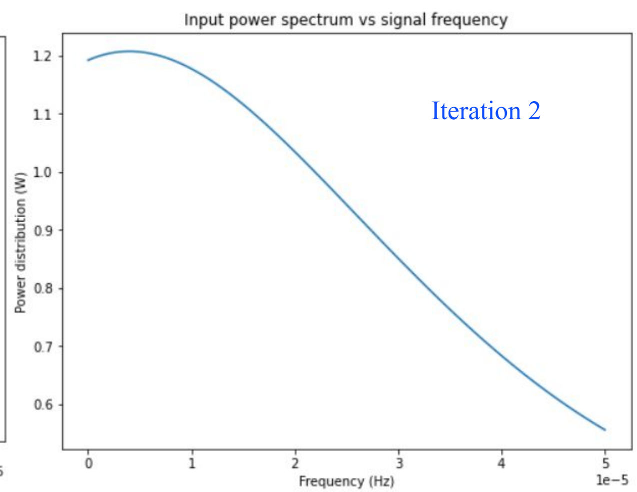
Implement: $\chi_{i+1}(\Omega_k) = \chi_i(\Omega_k) \frac{\nu(\chi_i, \Omega_k)}{n_\theta}$

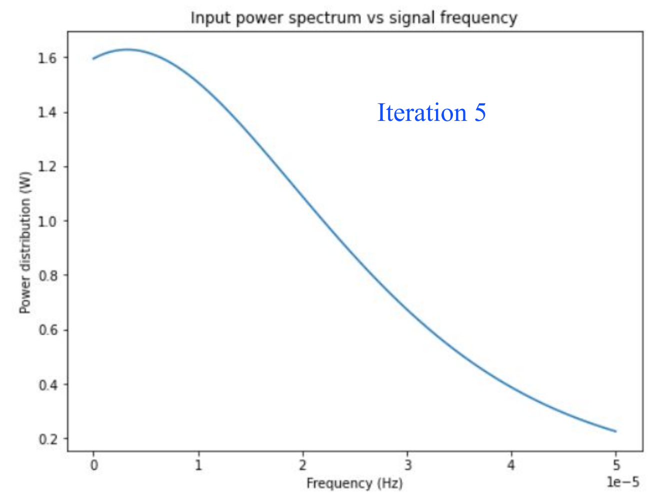
No. of parameters

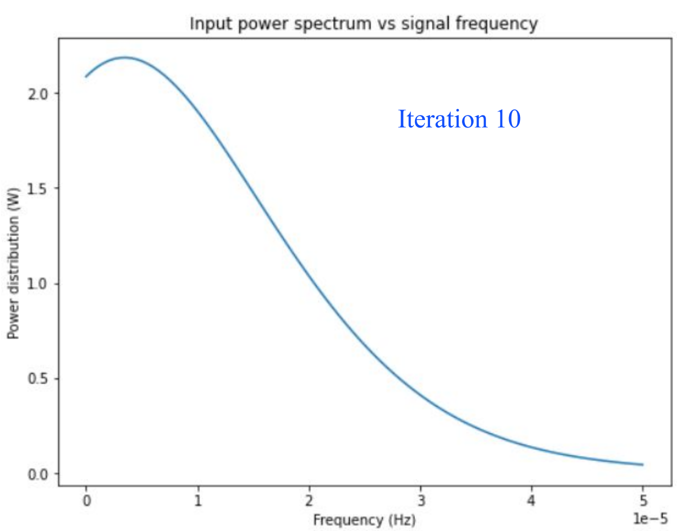
If $\max(\nu(\chi_i, \Omega_k) - n_\theta) < \epsilon$ then terminate, otherwise return to Step 2

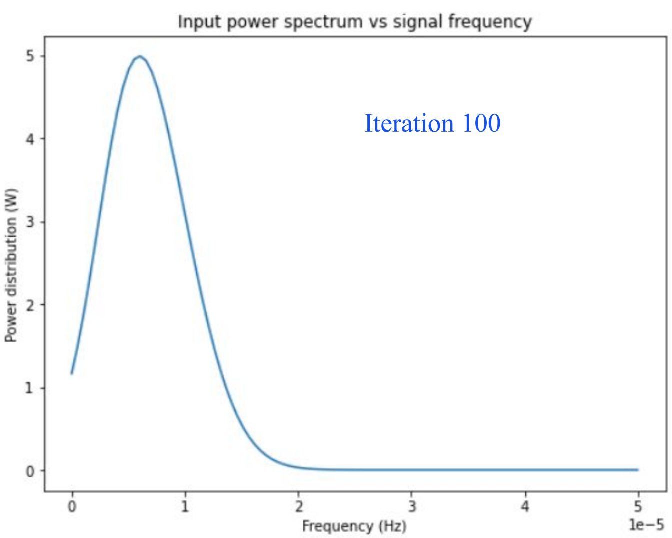
Power Spectrum Optimization



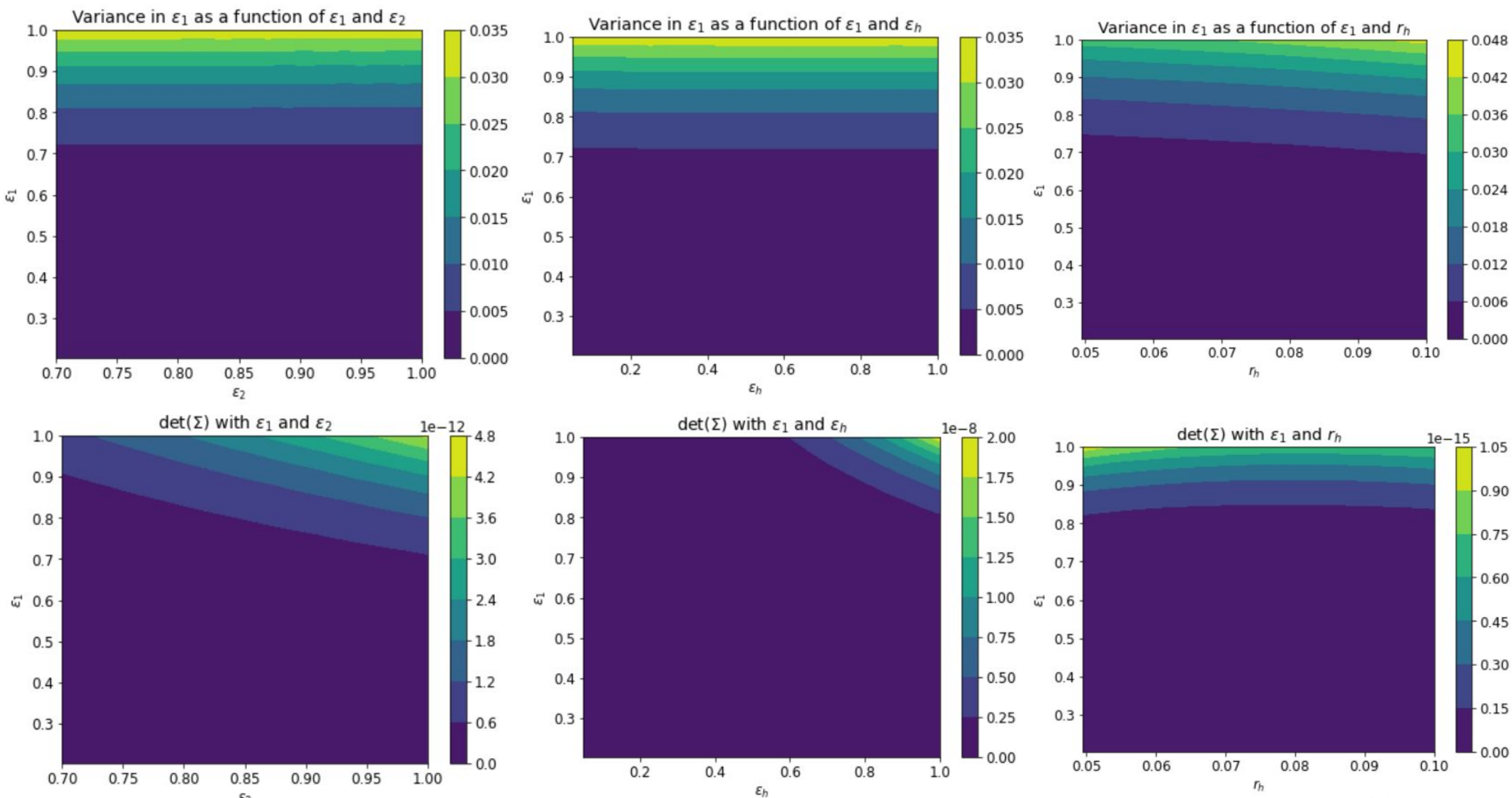








Optimal Experimental Configuration



Future Work

- Understand the mathematics of the Power Spectrum Optimization algorithm. Answer questions like: What is being minimized? Why are the answers not exactly equal to the Optimal Control Frequency analysis?
- Intuitively explain trends observed in Optimal Experimental Configuration
- Estimate or model the noise in our system from various sources (ambient fluctuations of temperature, measurement noise, etc.)
- Time-domain analysis of our original nonlinear system without assuming steady state conditions. Frequency domain is only for LTI systems. Present optimal frequencies are still impractical.
- A different problem can be posed: How do we get maximum information out of the system within a fixed amount of time?

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Acknowledgement and References

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References

[1] M. Constancio Jr, R. X. Adhikari, O. D. Aguiar, K. Arai, A. Markowitz, M. A. Okada, and C. C. Wipf, "Silicon emissivity as a function of temperature", International Journal of Heat and Mass Transfer **157**, 119863 (2020).

[2] E. D. Hall, "Fisher matrix methods for transfer function measurement", ligo (2015).

[3] R. Pintelon and J. Schoukens, *System identification: a frequency domain approach* (John Wiley & Sons, 2012).

[4] J. Tellinghuisen, "Statistical error propagation", The Journal of Physical Chemistry A **105**, 3917–3921 (2001).

[5] D. Wittman, "Fisher matrix for beginners", Technical report, UC Davis.

[6] Y. A. Çengel, *Heat transfer: a practical approach*, McGraw-Hill series in mechanical engineering (McGraw-Hill, 2003) Chap. 12.