Calculations for optimal mass distribution on multi stage suspensions

Edgard Bonilla

January 2019

1 Summary of results

1.1 Mass distribution:

We study the mass distribution that minimizes the high frequency[1] longitudinal transmission down an N-stage pendulum ($N \geq 2$), given a fixed total payload $P = \sum_{n=1}^{N} m_n$ and given the mass of the last stage m_N.

The result is given in terms of the (N-1)-th root of the test-mass-to-payload ratio $\rho = \left(\frac{m_N}{P}\right)^{\frac{1}{N-1}}$ by:

$$m_n = (1-\rho)\,\rho^{n-1} P \quad ; \quad n = 1, 2, ..., N-1 \tag{1}$$

Where m_n represents the mass of each stage of the suspension.

The simplest interpretation for this result is that in the optimal distribution, the payloads of successive stages of the suspension are in geometric progression with the ratio ρ, which interpolates between the total payload and the mass of the last stage:

$$\sum_{n=j}^{N} m_n = \rho^{j-1} P \tag{2}$$

This condition is illustrated in Figure 1, where the bold colors represent the stages considered in the sum (2).

1.2 Longitudinal transmission:

The asymptotic value for the longitudinal transmission of an N-stage pendulum is:

$$\frac{x_N}{x_g} \approx \left(\frac{g}{(2\pi f)^2}\right)^N \frac{1}{\prod\limits_{k=1}^{N} L_k} \frac{\prod\limits_{k=1}^{N}\left(\sum\limits_{n=k}^{N} m_n\right)}{\prod\limits_{k=1}^{N} m_k} \tag{3}$$

When the optimal mass distribution is used, the transmission is:

$$\frac{x_N}{x_g} \approx (-1)^N \left(\frac{g}{(2\pi f)^2}\right)^N \frac{1}{\prod\limits_{k=1}^{N} L_k} \frac{1}{(1-\rho)^{N-1}} \tag{4}$$

[1] The term 'high frequency' refers to the inertial range of frequencies for the transfer function, where the frequencies are much larger than the highest frequency mode $f_{\max}$.

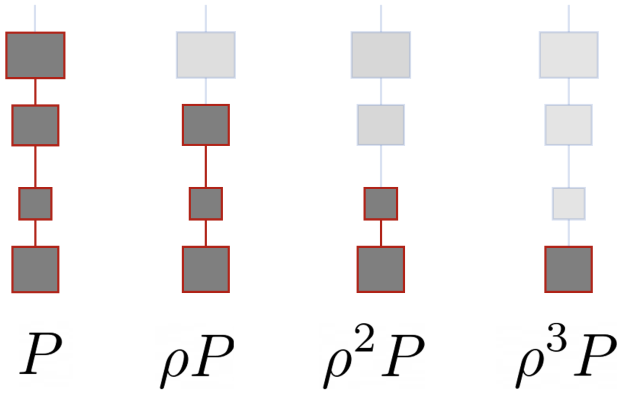

Figure 1: Illustration of the optimal mass distribution for a quadruple suspension. The minimum longitudinal transmission is found when the successive payloads are the same ratio from one another.

L_k represents the length of a segment of suspension connecting two masses, ending at mass m_k, g is the acceleration of gravity and f is the frequency, which is assumed to be higher than the highest resonant frequency $f_{\max}$ of the multi stage pendulum.

1.2.1 Range of applicability:

In the case of the optimal mass distribution, $f_{\max}$ is bounded above by:

$$2\pi f_{\max} \leq \frac{1+\sqrt{\rho}}{\sqrt{1-\rho}}\sqrt{\frac{g}{L_{\min}}} \tag{5}$$

Where $L_{\min}$ is the minimum length in the multi-stage pendulum.

We conclude that the approximation (4) is valid for $f \gg f_{\max}$.

2 Derivation of the solution

We start with the asymptotic equation for the transmission to the last stage of a N-stage, 1-D pendulum:

$$C_1 = \left(\frac{g}{(2\pi f)^2}\right)^N \frac{1}{\prod\limits_{k=1}^{N} L_k} \frac{\prod\limits_{k=1}^{N}\left(\sum\limits_{n=k}^{N} m_n\right)}{\prod\limits_{k=1}^{N} m_k} \tag{6}$$

We seek to minimize this transmission, with two constraints: the total payload P (sum of all masses) is fixed and so is the mass m_N of the last stage. The payload constraint can be written as:

$$G = \sum_{n=1}^{N} m_n - P = 0 \tag{7}$$

2.1 Case $N = 2$

In this case, the two constraints impose values for both masses. $m_1 = P - m_2$. And the optimal transmission is given by:

$$C_1^* = \left(\frac{g}{(2\pi f)^2}\right)\frac{1}{(L_1 L_2)}\frac{1}{1-\frac{m_2}{P}} \tag{8}$$

2.2 Case $N \geq 3$

In the general case with more than 2 masses, we can attempt to minimize C_1 within the constraint imposed by G by introducing a Lagrange multiplier λ and attempting to minimize:

$$F = C_1 + \lambda G \tag{9}$$

The equations that must be satisfied by the optimal masses can be found by taking the partial derivatives with respect to each m_j, $j = 1, 2, ..., N-1$.

$$\frac{\partial F}{\partial m_j} = C_1\left(\sum_{k=1}^{j}\frac{1}{\sum\limits_{n=k}^{N} m_n} - \frac{1}{m_j}\right) + \lambda = 0 \quad , \quad j = 1, 2, ..., N-1 \tag{10}$$

Where the first term is easily obtained by taking the logarithmic derivative of C_1.

We can eliminate the constant λ from all of these equations by comparing adjacent masses:

$$\frac{\partial F}{\partial m_j} - \frac{\partial F}{\partial m_{j+1}} = C_1\left(\sum_{k=1}^{j}\frac{1}{\sum\limits_{n=k}^{N} m_n} - \frac{1}{m_j} - \sum_{k=1}^{j+1}\frac{1}{\sum\limits_{n=k}^{N} m_n} + \frac{1}{m_{j+1}}\right) = 0 \tag{11}$$

Note that all terms on the big sum cancel, except for the one with $k = j+1$, which yields:

$$C_1\left(-\frac{1}{\sum\limits_{n=j+1}^{N} m_n} - \frac{1}{m_j} + \frac{1}{m_{j+1}}\right) = 0 \tag{12}$$

Finally, since m_N is set to be a constant different from zero and all masses must be positive, $C_1 \neq 0$. Which automatically sets the recursion relation for the m_k:

$$\left(\frac{1}{m_{j+1}} - \frac{1}{m_j}\right)\left(\sum_{n=j+1}^{N} m_n\right) = 1 \quad , \quad j = 1, 2, ..., N-2 \tag{13}$$

This is a nonlinear difference equation. Through a comparison with a continuous version of it, we deduce that the solution is an exponential function. To show this, set the optimal masses to be:

$$m_j = Ae^{-Bj} \quad , \quad j = 1, 2, ..., N-1 \tag{14}$$

with A and B constants to be determined. As a result, the first parenthesis in equation 13 is:

$$\left(\frac{1}{m_{j+1}} - \frac{1}{m_j}\right) = \frac{1}{m_{j+1}}\left(1 - e^{-B}\right) \tag{15}$$

As for the sum,we need to separate the term m_N since it is fixed:

$$\sum_{n=j+1}^{N} m_n = \sum_{n=j+1}^{N-1} Ae^{-Bn} + m_N = A\frac{e^{-B(j+1)} - e^{-BN}}{1 - e^{-B}} + m_N \tag{16}$$

$$\sum_{n=j+1}^{N} m_n = \frac{m_{j+1}}{1 - e^{-B}} + \left(m_N - \frac{Ae^{-BN}}{1 - e^{-B}}\right) \tag{17}$$

From equations 16 and 17 it can be inferred immediately that the exponential function is a valid solution to the recursion relation 13, as long as we choose A, B such that:

$$\frac{Ae^{-BN}}{1 - e^{-B}} = m_N \tag{18}$$

Finally, the other boundary condition for the problem is the fixed payload:

$$P = \sum_{n=1}^{N} m_n = \frac{Ae^{-B}}{1 - e^{-B}} \tag{19}$$

This allows us to solve for an unique set A, B as follows:

$$m_N = Pe^{-B(N-1)} \quad \Rightarrow \quad e^{-B} = \left(\frac{m_N}{P}\right)^{\frac{1}{N-1}} \tag{20}$$

$$A = \left(\left(\frac{P}{m_N}\right)^{\frac{1}{N-1}} - 1\right) P \tag{21}$$

Plugging these results back into the equation for m_j, we find the optimal mass distribution to be:

$$m_j = \left(\left(\frac{P}{m_N}\right)^{\frac{1}{N-1}} - 1\right)\left(\frac{P}{m_N}\right)^{\frac{-j}{N-1}} P \quad ; \quad j = 1, 2, ..., N-1 \tag{22}$$

similarly, the mass sum is given by:

$$\sum_{n=j}^{N} m_n = \frac{m_j}{1 - \left(\frac{m_N}{P}\right)^{\frac{1}{N-1}}} \quad ; \quad j = 1, 2, ..., N-1 \tag{23}$$

With this, we can find the minimum of the asymptotic value for the longitudinal transmission to the last mass:

$$C_1^* = \left(\frac{g}{(2\pi f)^2}\right)^N \frac{1}{\prod_{k=1}^{N} L_k} \frac{1}{\left(1 - \left(\frac{m_N}{P}\right)^{\frac{1}{N-1}}\right)^{N-1}} \tag{24}$$

Equations 22 and 24 extend to the case when $N = 2$, despite their derivation requiring $N \geq 3$. This can be verified by plugging values directly into the equations.

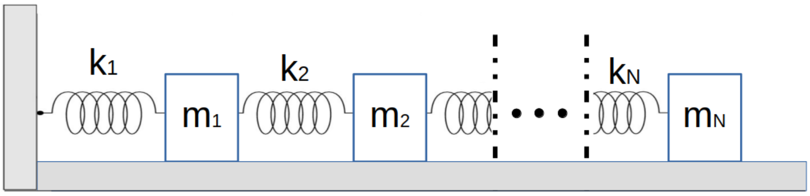

Figure 2: 1-D Multiple spring-mass system

3 Derivation of the approximation C_1

A multi-stage mass spring system with masses $m_1, m_2, ..., m_N$ and spring constants $k_1, k_2, ..., k_N$, like the one depicted in figure 2 satisfies a differential equation that can be written as:

$$\mathbf{M}\frac{d^2}{dt^2}\tilde{\mathbf{x}} = -\mathbf{K}\tilde{\mathbf{x}} + \begin{bmatrix} k_1 \\ \mathbf{0_{(N-1)\times 1}} \end{bmatrix} x_g \tag{25}$$

Where $\tilde{\mathbf{x}}$ is a vector listing all of the displacements of the masses from the equilibrium position, $\mathbf{M}$ is a diagonal matrix listing all of the masses, and $\mathbf{K}$ is a $N \times N$ matrix that lists the springs connecting each the masses:

$$\mathbf{K} = \begin{pmatrix} k_1+k_2 & -k_2 & & & \\ -k_2 & k_2+k_3 & -k_3 & & \\ & -k_3 & \ddots & \ddots & \\ & & \ddots & \ddots & -k_N \\ & & & -k_N & k_N \end{pmatrix} \tag{26}$$

We can change this equation to frequency domain by means of a Fourier transform, defining $\mathbf{\Omega^2} := \mathbf{M^{-1}K}$, we have:

$$\tilde{\mathbf{x}}(\omega) = (\mathbf{M^{-1}K} - \omega^2\mathbf{I_{N\times N}})^{-1} \begin{bmatrix} k_1/m_1 \\ \mathbf{0_{(N-1)\times 1}} \end{bmatrix} x_g(\omega) \tag{27}$$

The transfer function we hope to minimize for high ω is given by $\frac{x_N(\omega)}{x_g(\omega)}$, which can be obtained by finding the entry of $(\mathbf{M^{-1}K} - \omega^2\mathbf{I_{N\times N}})^{-1}$ of index $N, 1$. This is:

$$\frac{x_N(\omega)}{x_g(\omega)} = \frac{(-1)^{N+1}}{\det(\mathbf{M^{-1}K} - \omega^2\mathbf{I_{N\times N}})} \det(\mathbf{C_{1N}})\frac{k_1}{m_1} \tag{28}$$

Where $\mathbf{C_{1N}}$ is the matrix obtained by eliminating the first row and Nth column of the matrix we want to invert. It can be directly inspected that $\mathbf{C_{1N}}$ is an upper triangular matrix, and so its determinant is the product of its diagonal elements.

$$\det(\mathbf{C_{1N}}) = (-1)^{N-1}\frac{k_2k_3\cdots k_N}{m_2m_3\cdots m_N} \tag{29}$$

Now, we estimate the determinant in the denominator of equation 28 by treating it as the product of its eigenvalues. If ω^2 is bigger than all of the eigenvalues of $\mathbf{M^{-1}K}$, that is, for frequencies bigger than the resonant frequencies of the system, the determinant can be approximated by:

$$\det\left(\mathbf{M^{-1}K} - \omega^2 \mathbf{I_{N\times N}}\right) \approx (-1)^N \omega^{2N} \tag{30}$$

Which allows us to approximate the transmission as:

$$\frac{x_N(\omega)}{x_g(\omega)} \approx (-1)^N \frac{1}{\omega^{2N}} \frac{\prod\limits_{j=1}^{N} k_j}{\prod\limits_{j=1}^{N} m_j} = (-1)^N \frac{\det\left(\mathbf{M^{-1}K}\right)}{\omega^{2N}} \tag{31}$$

In the case of a multiple pendulum, the stiffnesses k_j are all given by the sum of the weight of the masses down the chain over the length of that segment: $k_j = g\frac{\sum\limits_{n=j}^{N} m_n}{L_j}$

Plugging this value for the stiffnesses yields the result:

$$\frac{x_N(\omega)}{x_g(\omega)} \approx (-1)^N \left(\frac{g}{\omega^2}\right)^N \frac{1}{\prod\limits_{k=1}^{N} L_k} \frac{\prod\limits_{k=1}^{N} \left(\sum\limits_{n=k}^{N} m_n\right)}{\prod\limits_{k=1}^{N} m_k} = (-1)^N C_1 \tag{32}$$

4 Range of validity, approximate forms

In the previous section, we found an exact solution for the mass and length distributions that minimize the magnitude of the transfer function from the suspension point to the last mass at high frequencies. This section is devoted to find the range of applicability of the approximate expression.

We expect the approximation to be true, as long as ω is large enough compared with the largest resonant frequency To find the resonant frequencies of the system it is enough to take the square roots of the eigenvalues of the matrix:

$$\mathbf{\Omega^2} := \mathbf{M^{-1}K} \tag{33}$$

We know that the eigenvalues of $\mathbf{\Omega^2}$ are positive because of both $\mathbf{K}$ and $\mathbf{M}$ are positive definite. Moreover, we can write $\mathbf{\Omega^2}$ in a way that makes this explicit by noticing:

$$\mathbf{\Omega^2} := \mathbf{M^{-1}K} = \mathbf{M^{-1/2}}(\mathbf{M^{-1/2}KM^{-1/2}})\mathbf{M^{1/2}} = \mathbf{M^{-1/2}}\ \mathbf{V}\ \mathbf{M^{1/2}} \tag{34}$$

$\mathbf{V}$ is a (symmetric) positive definite matrix that shares the same eigenvalues of $\mathbf{\Omega^2}$ and it is the matrix we will study from now on. Note that:

$$\mathbf{V} = \begin{pmatrix} \frac{k_1+k_2}{m_1} & \frac{-k_2}{\sqrt{m_1 m_2}} & & & \\ \frac{-k_2}{\sqrt{m_1 m_2}} & \frac{k_2+k_3}{m_2} & \frac{-k_3}{\sqrt{m_2 m_3}} & & \\ & \frac{-k_3}{\sqrt{m_2 m_3}} & \ddots & \ddots & \\ & & \ddots & \ddots & \frac{-k_N}{\sqrt{m_{N-1} m_N}} \\ & & & \frac{-k_N}{\sqrt{m_{N-1} m_N}} & \frac{k_N}{m_N} \end{pmatrix} \tag{35}$$

For simplicity, let us define $\rho = \left(\frac{m_N}{P}\right)^{\frac{1}{N-1}}$. It can be shown that with the mass distribution given by equation (1) the non-zero matrix elements of $\mathbf{V}$ are:

$$V_{ii} = \left(\frac{1+\rho}{1-\rho}\right) \frac{\left(\frac{g}{L_i}\right) + \rho\left(\frac{g}{L_{i+1}}\right)}{1+\rho}, \quad i = 1, 1, \ldots, N-1 \tag{36}$$

With $V_{NN} = \frac{g}{L_N}$. And:

$$V_{i,i+1} = V_{i+1,i} = -\frac{g}{L_{i+1}} \frac{\sqrt{\rho}}{(1-\rho)} \quad i = 1, 1, \ldots, N-1 \tag{37}$$

With $V_{N-1,N} = V_{N,N-1} = -\frac{g}{L_N}\sqrt{\frac{\rho}{1-\rho}}$.

We can bound the highest eigenvalue by using the circle theorem (Gershgorin's circle theorem), which states that the eigenvalues for $\mathbf{V}$ are in intervals around its diagonal elements. The radius of the intervals is given by the sum of the off-diagonal elements of each row.

Since $0 < 1-\rho < 1$, and noting that the diagonal elements of $\mathbf{V}$ are like weighted averages between 'frequencies', it is straightforward to arrive at the conservative bound:

$$\omega_{\max} \leq \frac{1+\sqrt{\rho}}{\sqrt{1-\rho}}\sqrt{\frac{g}{L_{\min}}} \tag{38}$$

Where $L_{\min}$ is the minimum length between two masses in the multi-stage pendulum.

The bound is obtained by noting that, according to the circle theorem:

$$\omega_{\max}^2 \leq \max_i \left(V_{ii} + \sum_{j\neq i} |V_{ij}|\right) \leq \max_i (V_{ii}) + \max_i \left(\sum_{j\neq i} |V_{ij}|\right) \tag{39}$$

And by adding that the off-diagonal terms are almost all zero ($V_{ij} = 0$ if $|i-j| > 1$), we are left with:

$$\omega_{\max}^2 \leq \max_i (V_{ii}) + 2\max_i \left(|V_{i,i+1}|\right) \tag{40}$$

Which can be readily bound above by noting that for $N \geq 2$, since $0 < \rho < 1$ the following inequalities hold:

1. $V_{NN} = \frac{g}{L_N} < \frac{1+\rho}{1-\rho}\frac{g}{L_N}$.
2. $|V_{N-1,N}| = \frac{g}{L_N}\sqrt{\frac{\rho}{1-\rho}} < \frac{g}{L_N}\frac{\sqrt{\rho}}{(1-\rho)}$.

3. $\frac{g}{L_i} \leq \frac{g}{L_{\min}}$ for all i.

4. $\max_i V_{ii} \leq \frac{1+\rho}{1-\rho} \frac{g}{L_{\min}}$ (Consequence of 1. and 3.).

5. $\max_i V_{i,i+1} \leq \frac{g}{L_{\min}} \frac{\sqrt{\rho}}{(1-\rho)}$. (Consequence of 2. and 3.).

Substituting the last two inequalities into equation (40) yields:

$$\omega_{\max}^2 \leq \frac{g}{L_{\min}} \frac{1+2\sqrt{\rho}+\rho}{1-\rho} = \frac{g}{L_{\min}} \frac{(1+\sqrt{\rho})^2}{1-\rho} \tag{41}$$

As promised.