Demonstration that orbits far from a binary measure its total mass including binding energy in general relativity

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Consider a binary on a circular orbit with masses m_1 and m_2 (both masses are the ones measured for that body in isolation) with its centre of mass at the origin. We want to check that a test particle orbit far away from the binary measures the binary's mass including its Newtonian binding energy, as expected. Since we just need to obtain the Newtonian equations of motion for the test particle, we only need to consider the purely temporal (00) component of the metric—the other components will provide post-Newtonian corrections suppressed by at least one power of the particle's velocity divided by the speed of light. See, e.g., Sec. 5.4.2 in [1].

We also only need to consider the $1/r$ pieces of this metric component (where r is the distance from the binary's centre of mass to the field point), since they will dominate far from the binary. We can obtain this to Newtonian and leading post-Newtonian order (i.e., corrections to the Newtonian gravity result of the order of the magnitude of the binary's orbital velocity squared or equivalently its Newtonian potential) from Eq. (2.17) of [2] (which also gives higher-order terms). From that paper, we have $g_{00} = -1 + 2U$, where

$$U = \frac{m}{r}\left(1 - \frac{\mu}{2b}\right) + \frac{\mu}{r}\left[(\hat{\mathbf{n}}\cdot\mathbf{v})^2 - \frac{m}{b}(\hat{\mathbf{n}}\cdot\hat{\mathbf{b}})^2\right]. \tag{1}$$

Here we work in $G = c = 1$ units and do not show the remainders, for simplicity. Additionally, $m := m_1 + m_2$, $\mu := m_1 m_2/m$ is the reduced mass, b is the orbital separation, $\hat{\mathbf{n}}$ is the radial unit vector (from the origin to the field point), $\mathbf{v}$ is the relative velocity of the objects in the binary, and $\hat{\mathbf{b}}$ is the unit separation vector of the objects in the binary. Here $\mathbf{v}$ and $\hat{\mathbf{b}}$ depend on the retarded time $t - r$ (as does b, but it is constant to this order of approximation—there is not yet any gravitational wave damping). This expression is given in the post-Newtonian harmonic coordinate system and we have changed the notation slightly from [2].

It turns out (see, e.g., Sec. 5.4.2 in [1]) that U acts as a potential in the Newtonian equations of motion for a test particle moving in the spacetime described by this metric. That is, the acceleration of the test particle moving far away from the binary is $\boldsymbol{\nabla} U$. Thus, the test particle will measure a mass of $m[1 - \mu/(2b)] = m - m_1 m_2/(2b)$, as expected. This neglects the dot product terms, but that is appropriate to do here: The orbital period of the test particle will be much longer than the orbital period of the binary, so one can average over a period of the inner binary's orbit when computing the test particle's acceleration, and the two dot product terms cancel when orbit averaged, since the magnitude of the binary's orbital velocity is $(m/b)^{1/2}$ and $\hat{\mathbf{v}}$ and $\hat{\mathbf{b}}$ just differ by a rotation by $\pi/2$ in the binary's orbital plane. (One can check that this result from the averaging also holds if one only averages after computing $\boldsymbol{\nabla} U$.)

One can also obtain the same result from the Einstein-Infeld-Hoffmann equations. See, e.g., Eq. (4.8) in [3], which gives the acceleration of the test particle in this scenario, noting that the R variable there is the same as our r above. In this expression, assuming a circular orbit for the inner binary, the only terms that are of leading order ($1/R^2$) in $1/R$ and do not depend on the test particle's small velocity $\mathbf{V}$ are the first term of that equation, which gives the Newtonian acceleration associated with m ($M = m$ for this test particle case) and the second and third lines of the cross terms given in that reference's Eq. (4.10), which exactly give the other contributions discussed above.

Additionally, if there are any concerns about the statement in footnote 3 in [2] that there is a typo in the metric perturbation in the Will and Wiseman paper cited for the binary's metric (since the correction of this typo is what gives the binding energy correction to m), see Eq. (6.10.1) in [1] for another expression for the metric perturbation that requires no typo corrections. The total gravitational mass M is given in that reference's Eq. (6.10.4), and the regularized potential in Eq. (4.1.13).

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[1] E. Poisson, "Post-Newtonian theory for the common reader," `https://archive.org/details/Eric_Poisson__PostNewtonian_theory_for_the_common_reader/mode/2up`.
[2] K. Alvi, Phys. Rev. D, **61**, 124013 (2000), arXiv:gr-qc/9912113.
[3] C. M. Will, Phys. Rev. D, **89**, 044043 (2014), arXiv:1312.1289 [astro-ph.GA].