Coherent $\widetilde{\mathcal{F}}$-statistic on semi-coherent candidate $\widehat{\mathcal{F}}$

R. Prix
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Notes on "confirming signal model" with a coherent observation for given semi-coherent candidate, and comparison to rough analytic estimate used in the follow-up paper

I. INTRODUCTION

Consider a perfectly-matched semi-coherent search on a CW signal with parameters λ_{s}, using N segments of data of length ΔT. Assume this yields a maximum-likelihood candidate with summed statistic $\widehat{\mathcal{F}}$, defined as

$$\widehat{\mathcal{F}}(\lambda_{\mathrm{s}}) \equiv \sum_{k=1}^{N} \mathcal{F}_k(\lambda_{\mathrm{s}}) = N\,\overline{\mathcal{F}}(\lambda_{\mathrm{s}})\,, \tag{1}$$

where $\mathcal{F}_k$ is the $\mathcal{F}$-statistic in segment k, and $\overline{\mathcal{F}}$ is the average $\mathcal{F}$-statistic over the segments.

The semi-coherent statistic $2\widehat{\mathcal{F}}$ follows a χ^2-distribution with $4N$ degrees of freedom and non-centrality parameter $\widehat{\rho^2}$, which we write as

$$P\left(2\widehat{\mathcal{F}}\middle| N, \widehat{\rho^2}\right) = \chi^2_{4N}(2\widehat{\mathcal{F}}; \widehat{\rho^2})\,. \tag{2}$$

Next, assume we perform a fully-coherent $\widetilde{\mathcal{F}}$-statistic search on λ_{s} using an amount of data T_{coh}. The coherent statistic $2\widetilde{\mathcal{F}}$ follows a χ^2-distribution with 4 degrees of freedom and non-centrality parameter $\widetilde{\rho^2}$, which we write as

$$P\left(2\widetilde{\mathcal{F}}\middle| \widetilde{\rho^2}\right) = \chi^2_{4}(2\widetilde{\mathcal{F}};\, \widetilde{\rho^2})\,. \tag{3}$$

For stationary noise, the squared-SNR ρ^2 for a signal scales linearly with the total amount of data used in the observation. If we know $\widehat{\rho^2}$, we can therefore deduce the non-centrality parameter of the coherent $\widetilde{\mathcal{F}}$-statistic as

$$\widetilde{\rho^2} = \kappa\,\widehat{\rho^2}\,, \tag{4}$$

where we defined

$$\kappa \equiv \frac{T_{\mathrm{coh}}}{N\,\Delta T}\,, \tag{5}$$

which is simply the fraction of the semi-coherent amount of data used for the coherent search.

Question: If we only know $2\widehat{\mathcal{F}}$, but not $\widehat{\rho^2}$, and we measure the perfectly-matched coherent statistic $2\widetilde{\mathcal{F}}(\lambda_{\mathrm{s}})$ using a fraction κ of the semi-coherent data, what is the probability distribution for $2\widetilde{\mathcal{F}}$? Or in other words, what is $P\left(2\widetilde{\mathcal{F}}\middle| 2\widehat{\mathcal{F}}, N, \kappa\right)$?

II. THE 'NAIVE' ESTIMATE

The 'naive' estimate for the range of expected coherent $2\widetilde{\mathcal{F}}$ values for given $2\widehat{\mathcal{F}}$ used in [1] and in the present paper [Eqs.(26),(30)] corresponds to assuming that $2\widehat{\mathcal{F}}$ fully determines $\widehat{\rho^2}$, using the *guess*:

$$\widehat{\rho^2}_o \equiv 2\widehat{\mathcal{F}} - 4N\,, \tag{6}$$

which therefore also specifies the coherent non-centrality $\widetilde{\rho^2}$ as

$$\widetilde{\rho^2}_o = \kappa\,\widehat{\rho^2}_o\,. \tag{7}$$

Furthermore, replacing the χ^2_4-distribution of Eq. (3) by a Gaussian distribution (which is a good approximation for $\widetilde{\rho}^2 \gg 1$), a "confidence band" of $\pm n_u$ 'standard deviations' is given around the mean of the distribution, i.e.

$$\mathrm{E}\left[2\widetilde{\mathcal{F}}\right] \approx 4 + \widetilde{\rho}^2{}_o \equiv 2\widetilde{\mathcal{F}}_o \tag{8}$$

$$\mathrm{var}\left[2\widetilde{\mathcal{F}}\right] \approx 2(4 + 2\widetilde{\rho}^2{}_o) \equiv \widetilde{\sigma}_o^2\,, \tag{9}$$

yielding the criterion

$$\left|2\widetilde{\mathcal{F}} - 2\widetilde{\mathcal{F}}_o\right| \leq n_u\,\widetilde{\sigma}_o\,, \tag{10}$$

for "signal-model confirmation" with n_u "sigma".

The problem with this simple criterion is that the "standard deviation" $\widetilde{\sigma}_o$ refers to the case of *known* $\widetilde{\rho}^2$, while the uncertainty on the actual non-centrality parameter for given $2\widehat{\mathcal{F}}$ will tend to widen the probability distribution for $2\widetilde{\mathcal{F}}$. Therefore it is unclear what numerical confidence corresponds to a given number n_u of "sigmas" $\widetilde{\sigma}_o$.

III. EXACT DERIVATION OF $P\left(2\widetilde{\mathcal{F}}\middle|\,2\widehat{\mathcal{F}}, N, \kappa\right)$

A. Non-centrality $\widehat{\rho}^2$ for given semi-coherent $2\widehat{\mathcal{F}}$

For given $2\widehat{\mathcal{F}}$, we can obtain the probability distribution for the non-centrality $\widehat{\rho}^2$ as

$$P\left(\widehat{\rho}^2\middle|\,2\widehat{\mathcal{F}}, N\right) = P\left(2\widehat{\mathcal{F}}\middle|\,N, \widehat{\rho}^2\right)\,\frac{P\left(\widehat{\rho}^2\middle|\,N\right)}{P\left(2\widehat{\mathcal{F}}\middle|\,N\right)} \tag{11}$$

$$\propto \chi^2_{4N}(2\widehat{\mathcal{F}}; \widehat{\rho}^2)\,P\left(\widehat{\rho}^2\middle|\,N\right)\,. \tag{12}$$

For injections the $\widehat{\rho}^2$-prior depends on the distributions used for the signal parameters. For candidates observed in real data, we can either assume a simple flat prior, or a Jeffreys' prior, i.e.

$$P\left(\widehat{\rho}^2\middle|\,N\right) \propto \begin{cases} \text{const.} \\ 1/\widehat{\rho}^2 \end{cases}\,. \tag{13}$$

B. Coherent $2\widetilde{\mathcal{F}}$ for given semi-coherent $2\widehat{\mathcal{F}}$

For given non-centrality $\widetilde{\rho}^2$, the probability distribution for $2\widetilde{\mathcal{F}}$ is simply given by Eq. (3). If instead we only know the semi-coherent statistic $2\widehat{\mathcal{F}}$, then we only know the probability distribution for $\widetilde{\rho}^2$, from Eq. (11), namely

$$P\left(\widetilde{\rho}^2\middle|\,2\widehat{\mathcal{F}}, N, \kappa\right) \propto P\left(\widehat{\rho}^2 = \widetilde{\rho}^2/\kappa\middle|\,2\widehat{\mathcal{F}}, N\right) \tag{14}$$

$$\propto \chi^2_{4N}\left(2\widehat{\mathcal{F}}; \widetilde{\rho}^2/\kappa\right)\,P\left(\widetilde{\rho}^2\middle|\,N\right)\,. \tag{15}$$

We can therefore compute the probability

$$P\left(2\widetilde{\mathcal{F}}\middle|\,2\widehat{\mathcal{F}}, N, \kappa\right) = \int_0^\infty P\left(2\widetilde{\mathcal{F}},\,\widetilde{\rho}^2\middle|\,2\widehat{\mathcal{F}}, N, \kappa\right)\,d\widetilde{\rho}^2 \tag{16}$$

$$= \int_0^\infty P\left(2\widetilde{\mathcal{F}}\middle|\,\widetilde{\rho}^2;\,2\widehat{\mathcal{F}}, N, \kappa\right)\,P\left(\widetilde{\rho}^2\middle|\,2\widehat{\mathcal{F}}, N, \kappa\right)\,d\widetilde{\rho}^2\,, \tag{17}$$

corresponding to marginalization over $\widetilde{\rho}^2$ for given semi-coherent $2\widehat{\mathcal{F}}$.

This last expression shows a subtle point: for given coherent non-centrality $\widetilde{\rho}^2$ *and* the semi-coherent $2\widehat{\mathcal{F}}$ over N segments, the probability distribution for $2\widetilde{\mathcal{F}}$ is *not necessarily* given by just the χ^2_4-distribution of Eq. (3). Namely, if one uses *the same data* to compute $2\widetilde{\mathcal{F}}$ and $2\widehat{\mathcal{F}}$ on, then for given $2\widehat{\mathcal{F}}$ we have more information about $2\widetilde{\mathcal{F}}$ than just $\widetilde{\rho}^2$, in other words $2\widetilde{\mathcal{F}}$ will not be independent of $2\widehat{\mathcal{F}}$ for given $\widetilde{\rho}^2$. This corresponds to the situation that was referred to as "data recycling mode" in [2]. On the other hand, if $2\widetilde{\mathcal{F}}$ is computed on independent data from that of $2\widehat{\mathcal{F}}$, then $2\widetilde{\mathcal{F}}$ *is* independent of $2\widehat{\mathcal{F}}$ for given $\widetilde{\rho}^2$, and so only in this case we strictly have

$$P\left(2\widetilde{\mathcal{F}}\middle|\,\widetilde{\rho}^2;\,2\widehat{\mathcal{F}},N,\kappa\right)=P\left(2\widetilde{\mathcal{F}}\middle|\,\widetilde{\rho}^2\right)=\chi^2_4(2\widetilde{\mathcal{F}};\widetilde{\rho}^2)\,, \tag{18}$$

which corresponds to the situation referred to as "fresh data mode" in [2].

Note that assuming no correlation between $2\widetilde{\mathcal{F}}$ and $2\widehat{\mathcal{F}}$ for given $\widetilde{\rho}^2$ is less informative than specifying a correlation. Therefore this assumption is *conservative*, in the sense that the posterior pdf will be wider in this case. Given that it is nontrivial to determine what particular correlation is implied by having used the same data, we will therefore safely continue with the conservative assumption of the "fresh data mode", even if in practice the same data was used.

IV. ANALYTIC APPROXIMATION FOR $P\left(2\widetilde{\mathcal{F}}\middle|\,2\widehat{\mathcal{F}},N,\kappa\right)$

One can numerically integrate Eq. (16), but we can also derive a simple analytical approximation. Let us first rewrite the integral more explicitly as

$$P\left(2\widetilde{\mathcal{F}}\middle|\,2\widehat{\mathcal{F}},N,\kappa\right)\propto\int_0^\infty\chi^2_4(2\widetilde{\mathcal{F}};\,\rho^2)\,\chi^2_{4N}\left(2\widehat{\mathcal{F}};\,\rho^2/\kappa\right)\,P\left(\rho^2\middle|\,I\right)\,d\rho^2\,. \tag{19}$$

In the present discussion of coherently following-up semi-coherent candidates, we will practically always be in the regime of $N\gg1$ and $\widehat{\rho}^2\gg1$. This allows us to approximate *both* χ^2-distributions in the integral by Gaussian distributions, i.e. we will use

$$\chi^2_{4N}(2\mathcal{F};\,\rho^2)\approx\text{Gauss}\left(2\mathcal{F};\,e_N(\rho^2),\sigma_N(\rho^2)\right)\,, \tag{20}$$

with either $N=1$ or $N=N$, and where we defined the mean and variance of χ^2_{4N} as

$$e_N(\rho^2)\equiv\mathrm{E}\left[\chi^2_{4N}(2\mathcal{F};\rho^2)\right]=4N+\rho^2\,, \tag{21}$$

$$\sigma^2_N(\rho^2)\equiv\mathrm{var}\left[\chi^2_{4N}(2\mathcal{F};\rho^2)\right]=2(4N+2\rho^2)\,. \tag{22}$$

Therefore we can write the approximate integral now as

$$P\left(2\widetilde{\mathcal{F}}\middle|\,2\widehat{\mathcal{F}},N,\kappa\right)\propto\int_0^\infty\frac{P\left(\rho^2\middle|\,I\right)}{\sigma_1(\rho^2)\,\sigma_N(\rho^2/\kappa)}\exp\left[-\frac{1}{2}\left(\frac{(2\widetilde{\mathcal{F}}-4-\rho^2)^2}{\sigma_1^2(\rho^2)}+\frac{(\widetilde{\rho}^2{}_o-\rho^2)^2}{\kappa^2\sigma_N^2(\frac{\rho^2}{\kappa})}\right)\right]d\rho^2\,, \tag{23}$$

where we rediscover the quantity

$$\widetilde{\rho}^2{}_o=\kappa\left(2\widehat{\mathcal{F}}-4N\right)=\kappa\,N\left(2\overline{\mathcal{F}}-4\right)\,, \tag{24}$$

defined in Eq. (7).

In order to continue, let us assume a flat ρ^2-prior, i.e. $P\left(\rho^2\middle|\,I\right)\propto$ const. By symmetry, we expect the distribution Eq. (23) to peak at $2\widetilde{\mathcal{F}}=2\widetilde{\mathcal{F}}_o=4+\widetilde{\rho}^2{}_o$, and we therefore approximate the variances $\sigma^2(\rho^2)$ by their value at the peak, i.e. $\rho^2=\widetilde{\rho}^2{}_o$, so we use

$$\sigma_1^2(\rho^2)\approx2(4+2\widetilde{\rho}^2{}_o)\equiv\widetilde{\sigma}_o^2\,, \tag{25}$$

$$\sigma_N^2\left(\rho^2/\kappa\right)\approx2(4N+\widetilde{\rho}^2{}_o/\kappa)\equiv\widehat{\sigma}_o^2/\kappa^2\,. \tag{26}$$

Extending the lower integration boundary in Eq. (23) from 0 to $-\infty$, we now find a Gaussian integral, which we can solve as

$$P\left(2\widetilde{\mathcal{F}}\middle|\,2\widehat{\mathcal{F}},N,\kappa\right)\approx\frac{1}{\sqrt{2\pi}\,\sigma_m}\exp\left[-\frac{1}{2}\frac{\left(2\widetilde{\mathcal{F}}-2\widetilde{\mathcal{F}}_o\right)^2}{\sigma_m^2}\right]=\text{Gauss}\left(2\widetilde{\mathcal{F}};\,2\widetilde{\mathcal{F}}_o,\sigma_m\right)\,, \tag{27}$$

where we defined the 'modified' standard-deviation σ_m as

$$\sigma_m^2 \equiv \widetilde{\sigma}_o^2 + \widehat{\sigma}_o^2 = 2\left(4 + 2\widetilde{\rho^2}_o\right) + 2\kappa^2 \left(4N + 2\frac{\widetilde{\rho^2}_o}{\kappa}\right) . \tag{28}$$

Remarkably, this final distribution agrees rather well with the 'naive' estimate of Eq. (8)(9), *except* for a broader width σ_m. Compared to the 'naive' width $\widetilde{\sigma}_o$ of Eq. (9), σ_m contains an additional broadening $\widehat{\sigma}_o$ added in quadrature, which accounts for the uncertainty in the estimation of the non-centrality parameter $\widehat{\rho^2}$ of the semi-coherent statistic $2\widehat{\mathcal{F}}$.

V. NUMERICAL EXAMPLES

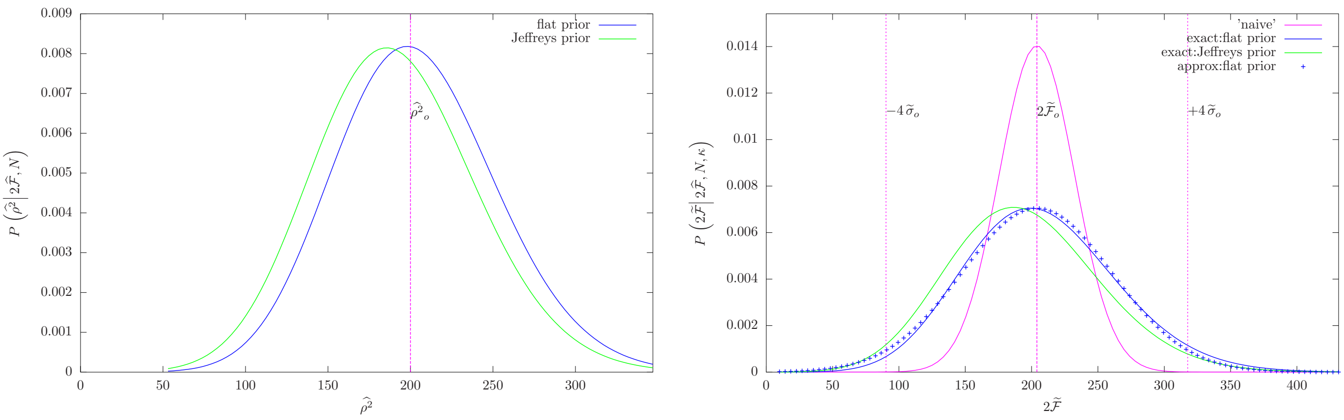

FIG. 1: PDFs for given $2\overline{\mathcal{F}} = 5.0$, $N = 200$, $\kappa = 1.00$: $2\widetilde{\mathcal{F}}_o = 204.0$, $\widetilde{\sigma}_o = 28.4$, and $\sigma_m = 56.6$.

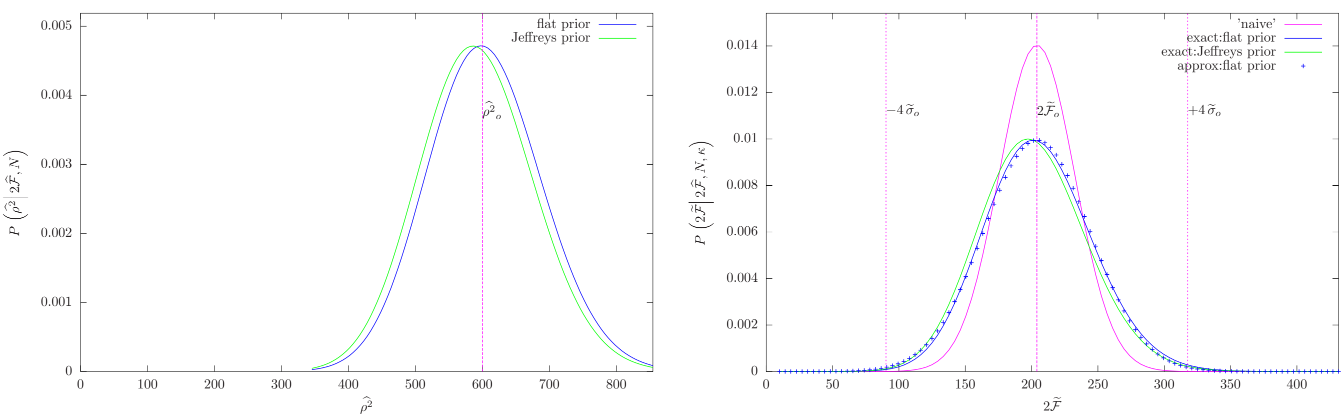

FIG. 2: PDFs given $2\overline{\mathcal{F}} = 5.0$, $N = 600$, $\kappa = 0.33$: $2\widetilde{\mathcal{F}}_o = 204.0$, $\widetilde{\sigma}_o = 28.4$, and $\sigma_m = 40.1$.

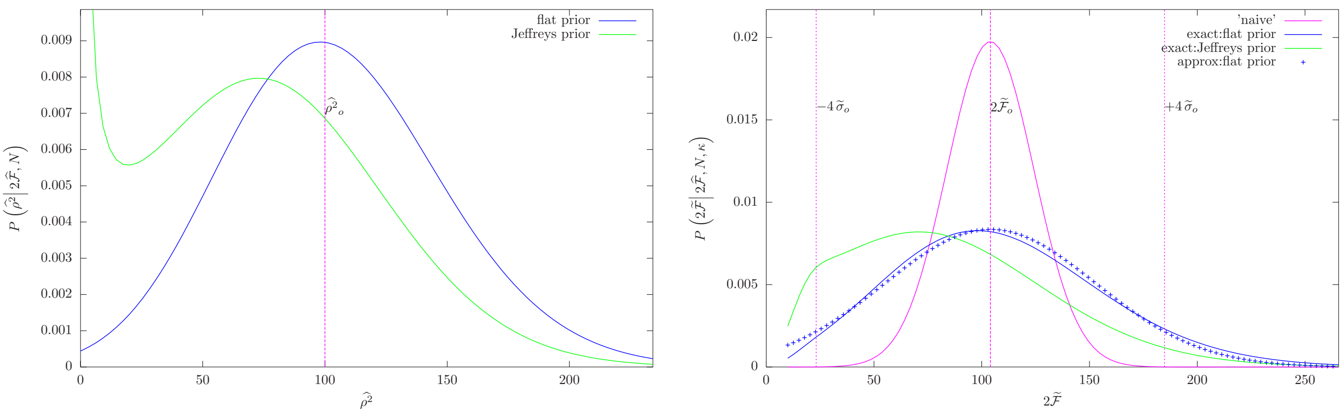

FIG. 3: PDFs for given $2\overline{\mathcal{F}} = 4.5$, $N = 200$, $\kappa = 1.00$: $2\widetilde{\mathcal{F}}_o = 104.0$, $\widetilde{\sigma}_o = 20.2$, and $\sigma_m = 49.1$.

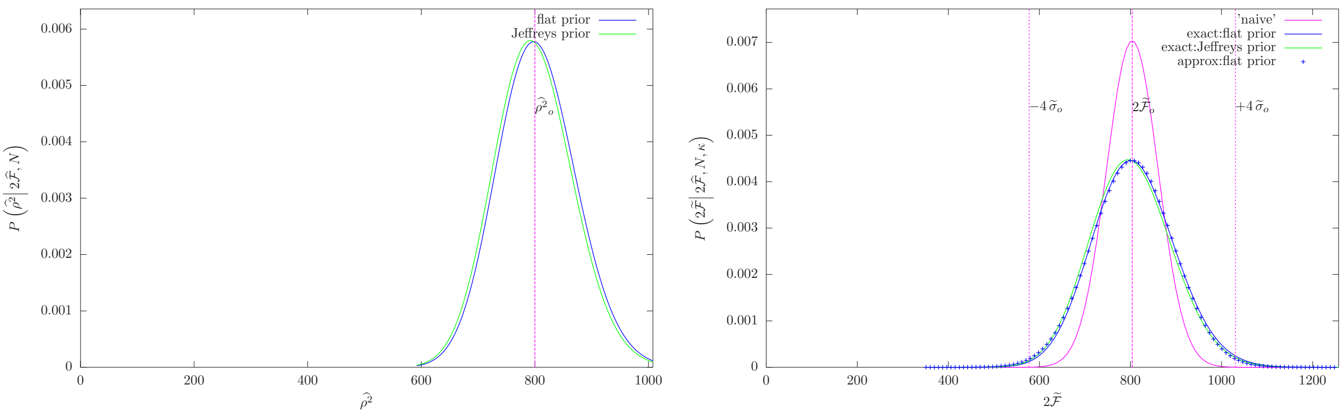

FIG. 4: PDFs for given $2\overline{\mathcal{F}} = 8.0$, $N = 200$, $\kappa = 1.00$: $2\widetilde{\mathcal{F}}_o = 804.0$, $\widetilde{\sigma}_o = 56.6$, and $\sigma_m = 89.5$.

[1] M. Shaltev, Journal of Physics Conference Series **363**, 012043 (2012), 1201.4656.
[2] C. Cutler, I. Gholami, and B. Krishnan, Phys. Rev. D. **72**, 042004 (2005), gr-qc/0505082.