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STATISTICAL METHODS OF GRAVITATIONAL-WAVE DETECTION AND ASTROPHYSICS

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1. INTRODUCTION

A century ago, Einstein and others laid down the foundation of what is now known as general relativity — the theory of how mass and the curvature of spacetime interact. One prediction of this theory is the radiation of energy through gravitational waves (Einstein 1916) from accelerating massive systems like astronomical binaries. While normally an immeasurable effect for widely separated, non-compact systems like the Solar System, close binaries with black hole or neutron star components emit potentially detectable and distinct gravitational-wave signatures. Detection by laser interferometers like the kilometer-scale instruments in Livingston, Louisiana and Hanford, Washington has been pursued because they are capable of detecting the diminutive effect of passing waves — in the detections discussed here, the length of the 4 km interferometer arms varied by only $\sim 10^{-18}$ m. Searches for gravitational waves from compact binaries use models which are parameterized by the physical properties of the binary (e.g., compact object masses, angular momenta, orientation and position; usually more than 15 parameters), which describe the effect of the wave impinging on a network of gravitational-wave detectors. To account for uncertainty in the models, or when a complete signal model is unavailable, more generic searches are employed, using methods that assume no particular signal morphology. In the autumn of 2015, the two LIGO interferometers (Abbott et al. 2016e) recorded several transient events from the mergers of binary black holes (Abbott et al. 2016b), confirming their existence and measuring their properties (Abbott et al. 2016a) using a variety of sophisticated statistical techniques. Abbott et al. (2016g) describes the basic, order-of-magnitude physics behind the radiated signal and its interpretation in the first detection.

2. TRANSIENT SEARCHES

The transient searches employed by LIGO can separated into those which assume a particular source model (e.g. searches for binary mergers, see Abbott et al. (2016b) and references therein) and more generic searches which only enforce signal consistency between instruments (see Allen et al. (2012) and references therein). Both types of searches use time-series filtering algorithms (Anderson et al. 2001; Allen et al. 2012) which produce lists of times and amplitudes relative to the noise (encoded in the signal-to-noise ratio or SNR) which characterize putative signals embedded in a noisy data stream. In addition to the SNR, modeled searches can also leverage a χ^2 goodness-of-fit test (Allen 2005), which tests against the expected distribution of signal power along the realized evolution of the signal. The main function of statistics like the SNR and χ^2 is to distinguish the event candidate from the non-astrophysical transient background distribution and establish its statistical significance. Thus, the final product of these searches is the probability under the null hypothesis of noise alone producing a test statistic value as large or larger than the candidate's (Capano et al. 2016; Abbott et al. 2016d).

The distribution of expected SNRs from compact binary mergers has a distinct power-law slope, but it is also overlapped — particularly at lower SNR — by a SNR distribution produced only from filtering fluctuations produced by Gaussian instrument noise. Ideally, the SNR distribution of the noise fluctuation events falls faster (exponentially versus power law) than our expected signal distribution. However, searches must also deal with transients in the data which are induced by the instrument and its environs (Abbott et al. 2016a) — colloquially called "glitches". This additional population imposes fatter tails on the idealized fluctuation-induced distribution.

In practice, there is no analytic description for the instrumental transient rate, and so all transient searches must develop the sample of the noise event rate empirically. Both the generic and modeled searches demand temporal coincidence of candidate events between the instruments. To form the background sample, the coincidence test is reapplied to events in one instrument shifted in time with respect to others. This samples the *accidental* coincidence rate which is built up over many different time-shifts. Any true gravitational-wave coincidence would be broken by this procedure because time-of-flight between the instrument sites is small compared to the slide duration. We further model the event times as Poissonian, and estimate the rate constant as a function of SNR from the list of slide events. We can then calculate the probability of obtaining event SNRs as loud as or louder than a given threshold.

3. PARAMETER ESTIMATION

Once observed by the searches, a posterior distribution over the parameters describing the source of a signal is sampled using forward-modeling Bayesian methods that demand explicit modeling of signal and noise alike. The likelihood function is formed from the residuals left after signal subtraction from the data time series. This likelihood, coupled with the priors on source parameters (e.g., uniform in compact object masses and spins, isotropic in orientation, uniform in volume in the local universe), provide the posterior probability density for the source parameters, which is sampled using stochastic sampling techniques, particularly Markov Chain Monte Carlo and nested sampling (Abbott et al. 2016a; Veitch et al. 2015). The noise in each detector is largely Gaussian and stationary on timescales relevant for transient signals. However, occasional non-Gaussian noise transients — transients which are terrestrial in origin and form the tails mentioned previously — occur independently in each instrument. Since compact binary models imply strong assumptions about signal morphology, the glitch component of the noise can typically be dismissed without introducing significant biases in parameter constraints.

However, for signals with fewer model constraints, glitches may resemble components of the signal, potentially resulting in false characterization if ignored. To account for the uncertainty in glitch behavior at the time of a signal, the non-Gaussian components of the noise (assumed uncorrelated between instruments) are modeled simultaneously with the signal (assumed correlated across detectors). Since neither the morphology of the signal nor the glitch(es) are known, both are modeled as a combination of Sine-Gaussian (*Morlet*) wavelets. The number of noise and signal wavelets in each detector is not fixed a priori, rather marginalized using a reverse-jump Markov Chain Monte Carlo (RJMCMC). This enables jumps between model spaces in addition to traditional MCMC jumps within a single model space, allowing for model comparison to be done "on the fly", and avoiding numerous fixed-dimension MCMC analyses and marginal likelihood estimates (Cornish & Littenberg 2015; Abbott et al. 2016c).

4. RATES AND MASS DISTRIBUTIONS

One of the key science outputs from these observations is the density of merging binary black holes in spacetime (the merger rate), expressed as a number of mergers per cubic gigaparsec (Gpc^3) per year[1]. To calculate this number, LIGO must calculate the number of detected mergers (the "numerator" in units of counts) and the spacetime volume that it has searched (the "denominator" in units of $\mathrm{Gpc}^3\,\mathrm{yr}^1$). There are statistical challenges to both calculations.

While two *confident* gravitational-wave detections were reported (GW150914 and GW151226) there was a third candidate identified with measured cumulative accidental coincidence (or false alarm) rate of $(2.3\,\mathrm{yr})^{-1}$ (Abbott et al. 2016d). This candidate (LVT151012) has a significance, or p-value, of 0.045 (Abbott et al. 2016b). To account for our uncertainty about the origin of this trigger we fit a mixture model comparing the SNR distributions of both the terrestrial and astrophysical populations to the set of triggers from our searches, including the three candidates and many more (Abbott et al. 2016b,i,h; Farr et al. 2015). From this model, we infer a posterior probability that the LVT151012 trigger is associated with an astrophysical source of 0.86. We use the posterior on the astrophysical population in the mixture model to infer coalescence rates.

In regards to the volume, the maximum distance of an observable source (and so also the spacetime volume) is strongly dependent on the assumed population of sources. This volume is calculated by measuring the sensitivity of the model-dependent searches to simulated binaries constructed from a source population model. Fitting a hierarchical power-law model, with $p\,(m_1) \propto m_1^{-\alpha}$, to our three candidates, accounting for our mass measurement uncertainties and selection effects (Loredo 2004; Mandel et al. 2016) yields a posterior median and 90% credible interval of $\alpha = 2.5^{+1.5}_{-1.6}$ (Abbott et al. 2016b); not surprisingly, given three possible detections, the source population is poorly constrained. To account for this uncertainty when estimating rates, we estimate the merger rate under three different population assumptions (Abbott et al. 2016i,h,b):

- The population of mergers is a three-component mixture containing binaries exactly like GW150914, GW151226, and LVT151012 (if this is, indeed, a merging BBH system). We then infer posterior medians and 90%

[1] One parsec is 3.1×10^{16} m, or about three light-years. There are about $12{,}000\,\mathrm{Gpc}^3$ in the observable universe. See Hogg (1999) for a discussion of distances and times in cosmology.

credible intervals for the mixing fractions (single-type merger rates) of $R_{\mathrm{GW150914}} = 3.4^{+8.8}_{-2.8}\,\mathrm{Gpc^{-3}\,yr^{-1}}$, $R_{\mathrm{GW151226}} = 36^{+95}_{-30}\,\mathrm{Gpc^{-3}\,yr^{-1}}$, and $R_{\mathrm{LVT151012}} = 9.1^{+31.0}_{-8.5}\,\mathrm{Gpc^{-3}\,yr^{-1}}$. The total BBH merger rate is inferred to be $R = 55^{+103}_{-41}\,\mathrm{Gpc^{-3}\,yr^{-1}}$ — the total rate is not a simple sum of the single-type rates because they are correlated.

- The population density of merging black holes is flat in the log of their masses, $p\,(m_1, m_2) \propto 1/\,(m_1 m_2)$. Under this assumption, which is consistent with our population inference, the merger rate is $R = 31^{+42}_{-21}\,\mathrm{Gpc^{-3}\,yr^{-1}}$.

- The population of merging binary black holes is a power law in the larger mass with a slope corresponding to the power-law (with slope of $\alpha = 2.35$) of the mass distribution of recently-formed stars (Salpeter 1955), conditional on the larger mass, flat in the smaller mass between $M_{\mathrm{min}} = 5\,M_\odot$ and m_1: $p\,(m_1, m_2) \propto m_1^{-\alpha}/\,(m_1 - M_{\mathrm{min}})$. Under this assumption, which is also consistent with our population inference, the merger rate is $R = 97^{+135}_{-67}\,\mathrm{Gpc^{-3}\,yr^{-1}}$.

Between the three different population assumptions, our 90% credible intervals on the rate span the range 9–240 $\mathrm{Gpc^{-3}\,yr^{-1}}$.

The posterior predictive distribution for the rate can be used to extrapolate the expected number of future detections. While dependent on progress with planned improvements in the instruments (Abbott et al. 2016e,c), using a reasonable range of estimates for the expected detector sensitivities in the upcoming six-month LIGO observing run to start in late 2016 (see Abbott et al. (2016f)), the probability of at least 10 more confident detections (like GW150914 and GW151226) is between 15–80%. The future looks bright for gravitational-wave astronomy.

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