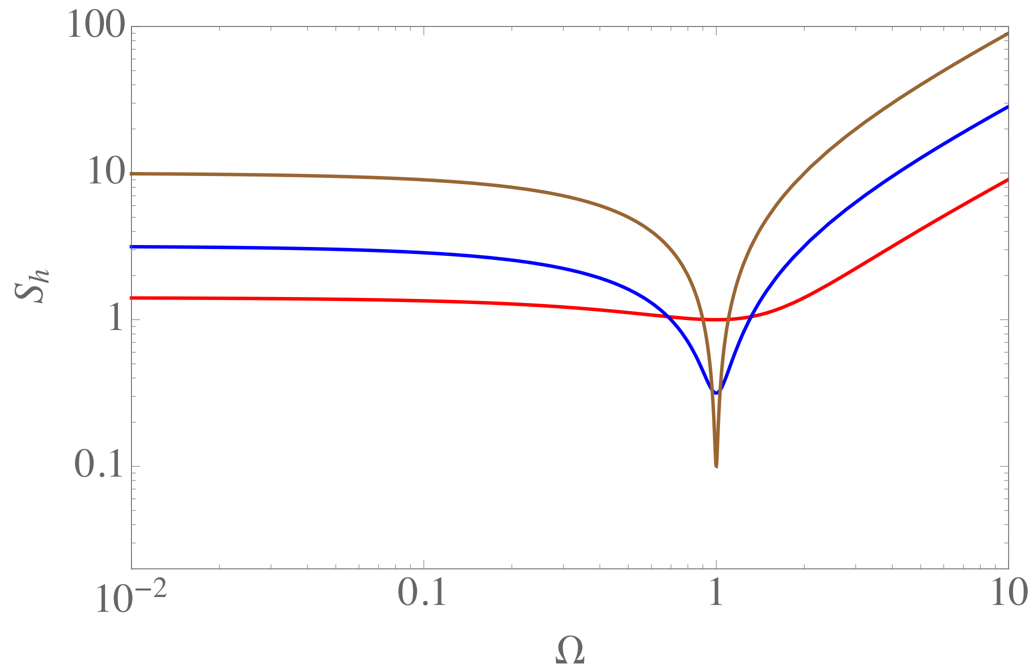
Quantum Cramer-Rao Bound

Belinda Pang and Yanbei Chen

[to be continued by Rana]

Caltech Relativity Theory (CaRT) Group

Mizuno Theorem & Energetic Quantum Limit



J. Mizuno's Ph.D. Thesis:

[Sensitivity Gain] * [Bandwidth]

is preserved.

$$\int \frac{1}{S_h(f)} df \propto P$$

Braginsky, Khalili, Gorodetsky & Thorne: **Energetic Quantum Limit** (arXiv:gr-qc/9907057)

$$\frac{S}{N} = \frac{4}{\hbar^2} \left\langle \left(\int_{-\infty}^{\infty} \Delta \mathcal{H}_I(t) dt \right)^2 \right\rangle$$

$$\frac{S}{N} = \frac{4}{\hbar^2 L^2} \int_{-\infty}^{\infty} |X_{signal}(\omega)|^2 S_{\mathcal{E}}(\omega) \frac{d\omega}{2\pi} \, , \quad \Rightarrow S_X(\omega) = \frac{\hbar^2 L^2}{4 S_{\mathcal{E}}(\omega)}$$

$S_{\mathcal{E}}$: energy fluctuations inside cavity

Connection to the Quantum Cramer-Rao Bound

system's density matrix depends on parameter θ $\quad \hat{\rho}(\theta): \quad i\frac{\partial\hat{\rho}}{\partial\theta} = \hat{L}\hat{\rho} - \hat{\rho}\hat{L}$ $\quad$ *L* is like interaction Hamiltonian!

X is an **unbiased** estimator $\quad \mathrm{tr}\left[\hat{X}\hat{\rho}(\theta)\right] = \theta$

minimum error $\quad \mathrm{tr}\left[\hat{\rho}(\hat{X}-\theta)^2\right] = \frac{1}{4\mathrm{tr}[\hat{\rho}\hat{L}^2]}$

Basically the same as Energetic Quantum Limit!

$$S_{hh}(\Omega) \geq \frac{\hbar^2 c^2}{4L_{\mathrm{arm}}^2 S_{PP}(\Omega)}$$

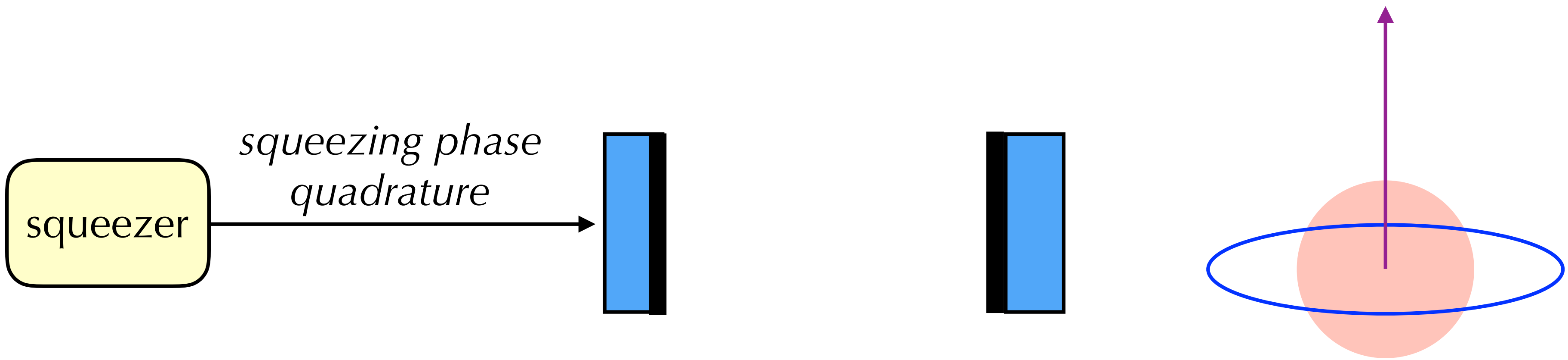
Tsang, Caves and Wiseman, PRL 2011

The QCRB requires high amplitude fluctuations

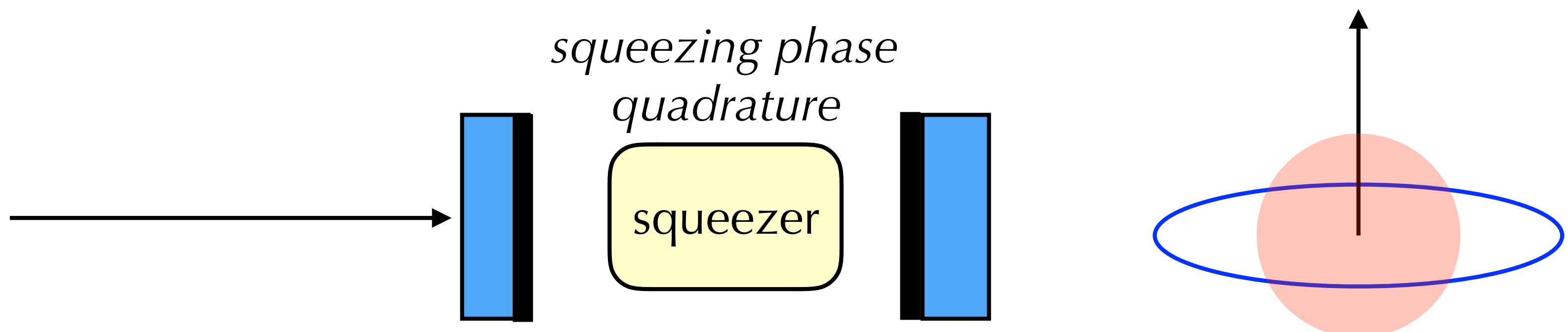
Does not require low phase fluctuations
[reaching it requires no optical losses]

- See **Rana**'s talk for more details.

Increasing the Cramer-Rao Bound (I)

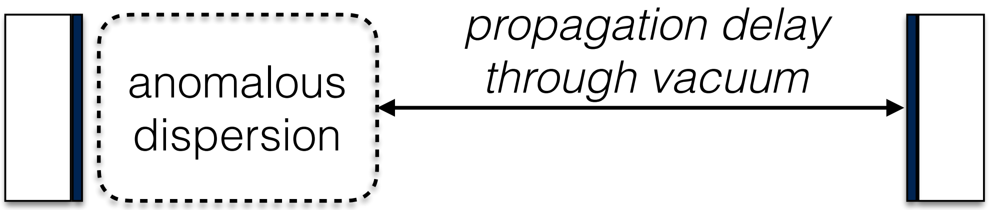


- By injection of squeezed vacuum
- Need to squeeze the signal quadrature, anti-squeeze the amplitude quadrature



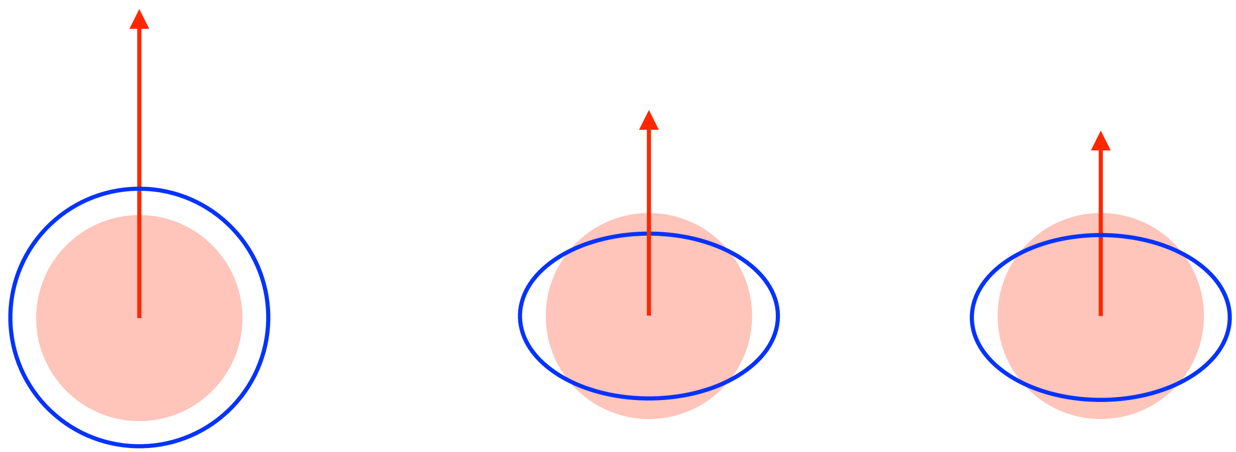
- By internal squeezing, **Mikhail Korobko**'s talk.

Increasing the Cramer-Rao Bound (II)



white light cavity
Wicht et al., S. Wise et al., Zhou, Shahriar et al.

- Phase Insensitive Amplification



signal	**amplified**	normal	squeezed
phase noise	anti-squeezed	squeezed	**squeezed**
amplitude noise	anti-squeezed	anti-squeezed	**anti-squeezed**

- Possible if the filter is **unstable** *[Ma, Miao, Zhao & Chen]*.
- Entire system can be stabilized.

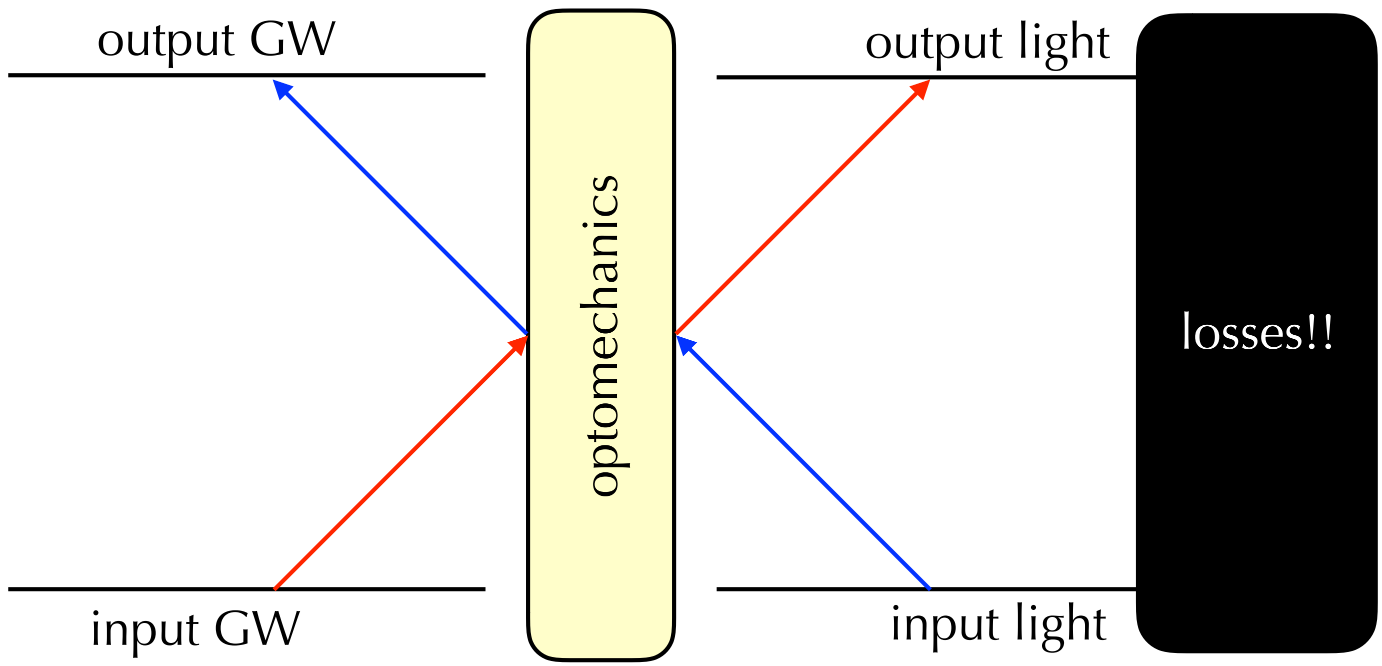
Increasing the Cramer-Rao Bound (III)

- Can optical spring (ponderomotive squeezing) be used to increase the Cramer-Rao bound?
- Is the CR bound reachable?

Rana and **Haixing**'s talk.

An interesting way to think about the QCRB

- Detector's emissivity of gravitational waves, when it is driven by vacuum fluctuations. *[Proposed by **Yuri Levin,** discussed further by **Smith-Lefebvre** and **Miao**].*



Conceptual Problem

is EM field radiating or test mass radiating?

related problem

does signal come from the motion of mirrors or phase shift of light?

Detector as Emitter

Using q-CRB, we can show that the best GW detector is also the best GW emitter when driven by quantum fluctuations (no classical drive)

Idea: higher SNR bound is achieved by increasing power fluctuations, which corresponds to higher probabilities of graviton emission

Energy radiation by gravitational waves

$$\frac{dE}{dt} = \frac{1}{5}\langle \dddot{I}_{ij} \dddot{I}^{ij} \rangle$$

$$I_{ij} = \int d^3x \, T^{00}(x_i x_j - \frac{1}{3}\delta_{ij} r^2)$$

power fluctuations → force noise → larger test mass motion

TT Gauge

Coordinates of particles moving along geodesics are constant in time, even when GW passes!

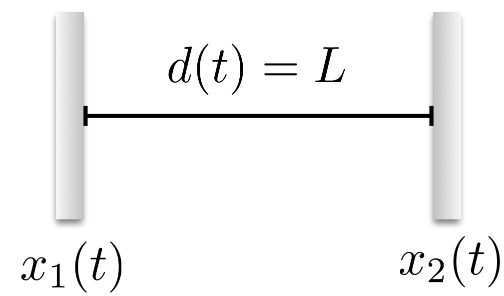
Metric $$ds^2 = -dt^2 + (1+h)dx^2 + (1-h)dy^2$$

geodesic equation → $$\frac{d^2x}{d\tau^2} = \frac{d^2x}{d\tau^2} = 0 \quad \text{for} \quad \vec{u}(0) = (1,0,0,0)$$

Coordinate and Proper distances

$$d(t) = x_1(t) - x_2(t) = L$$

$$D(t) = \int_{x_1}^{x_2} ds \approx \left(1 + \frac{1}{2}\boxed{h}\right) L$$



Hamiltonian in TT Gauge

Hamiltonian: $\hat{H} = \hat{H}_{\mathrm{cav}} + \hat{H}_{\mathrm{GW}} + \hat{H}_{\mathrm{int}}$

$$\hat{H}_{\mathrm{int}} = -G_h \hat{a}_1 \hat{h} \qquad \hat{h} = \int d\omega \sqrt{\frac{4G}{\omega}} [\hat{d}\omega e^{-i\omega t} + \hat{d}^\dagger \omega e^{i\omega t}]$$

Direct coupling of strain to cavity amplitude!

Derivable from action principle applying TT gauge constraints

Belinda Pang's slides, joint work by Yiqiu Ma & Belinda Pang

Probability of graviton emission

Calculate probability for the transition

$$|0\rangle_{\rm GW} \otimes |0\rangle_{\rm EM} \xrightarrow{\hat{H}_{\rm int}} |\Psi\rangle_{\rm GW} \otimes |f\rangle_{\rm EM}$$

sum over *f*

$$|\Psi\rangle_{\rm GW} = \int \sqrt{\frac{\omega}{4G}} h(\omega) \hat{d}^\dagger(\omega)|0\rangle$$

$$h(t) = \langle 0|\hat{h}(z=0,t)|\Psi\rangle$$
$$= \int h(\omega) e^{-i\omega t}\, d\omega$$

Probability of emission into waveform $h(\omega)$:

$$P = G_h^2 \int d\omega\, S_{a_1 a_1}(\omega) |h(\omega)|^2$$

$$P \propto SNR|_{\rm max}$$

Summary

- The quantum Cramer Rao bound is the fundamental limit to parameter estimation using a quantum probe (or using a quantum system to measure a classical signal)

- Increasing the bound on SNR to a particular waveform *h* means increasing power fluctuations inside the cavity

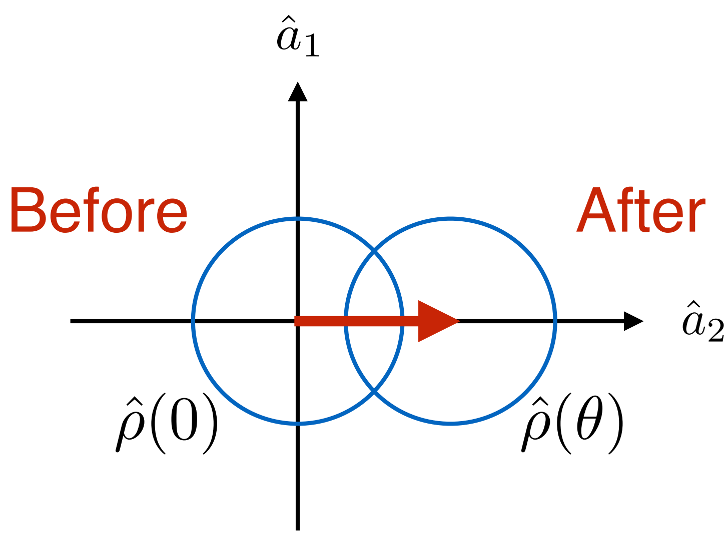
- Increasing SNR (or power fluctuations) also means we will increase GW radiation into the waveform *h*

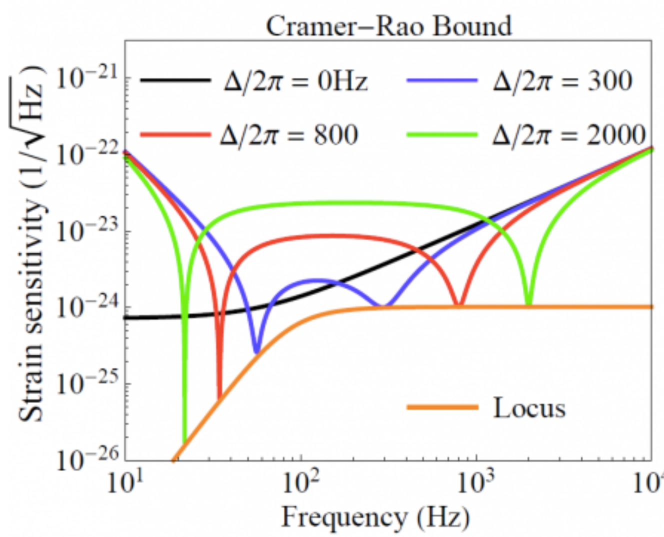
- This also means maximizing SNR for LIGO will also maximize GW radiation due to quantum fluctuations - possibly a fundamental source of quantum decoherence due to Heisenberg uncertainty! Best candidate for detecting such a decoherence?

Quantum Cramer Rao Bound

Fundamental limit derived from linear measurement theory

Idea: how distinguishable is the quantum state of a probe before and after detection of classical signal?





Miao, Haixing (2015)

$$\sigma_F(\Omega)|_{\min} = \frac{\hbar^2}{4|R_{xx}(\Omega)|^2 S_{FF}(\Omega)}$$

tuned cavity

$$SNR|_{\text{optimal}} \propto \int |h(\omega)|^2 S_{pp}(\omega)\, d\omega$$