Continuous Wave Data Analysis: Fully Coherent Methods

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1 Signal Model

1.1 GWs from rotating neutron star

We showed yesterday that the gravitational wave signal arriving at a time τ from an object a distance d away rotating with angular frequency Ω is

$$\overleftrightarrow{h}(\tau) = A_+ \cos[\Phi(\tau) + \phi_0]\, \overleftrightarrow{e}_+ + A_\times \sin[\Phi(\tau) + \phi_0]\, \overleftrightarrow{e}_\times \tag{1.1}$$

where

$$A_+ = h_0 \frac{1 + \cos^2 \iota}{2} \tag{1.2a}$$

$$A_\times = h_0 \cos \iota \tag{1.2b}$$

where the GW amplitude is

$$h_0 = \frac{4G\Omega^2 (I_1 - I_2)}{c^4 d} \tag{1.3}$$

and the phase evolution is

$$\Phi(\tau) = 2\Omega(\tau - \tau_0) \tag{1.4}$$

Some notes:

- I_1, I_2 and I_3 are the moments of inertia about the principal axes. Rotational oblateness means that $I_3 > I_1, I_2$. For a perfect spheroid, I_1 and I_2 would be equal and there would be no gravitational radiation. To get a "triaxial" neutron star, you need some sort of deformation.

- The signal was derived for an object rotating at a constant velocity. Since real neutron stars generally spin down gradually, we can modify this by Taylor expanding the phase.

$$\Phi(\tau) = f_0(\tau - \tau_0) + \frac{1}{2} f_1 (\tau - \tau_0)^2 + \frac{1}{3!} f_2 (\tau - \tau_0)^3 + \ldots \tag{1.5}$$

The frequency f_0 and spindowns f_1, f_2, etc are parameters of the signal.

- The time τ is the time that the waves arrive at some inertial point, usually taken to be the solar system barycenter (SSB). Since we're interested in the signal that arrives at time t at the location of the detector, we have to use the time delay

$$t = \tau + \frac{\vec{k} \cdot (\vec{r}_{\text{det}} - \vec{r}_{\text{ssb}})}{c} + \text{relativistic corrections} \tag{1.6}$$

which translates into a Doppler shift which depends on the sky position, as represented by the propagation direction $\vec{k}$.

- The signal may also be Doppler-shifted by the proper motion of the neutron star. If this motion is inertial (constant velocity), this is just a constant Doppler shift which we fold into the definition of f_0. If the neutron star is in a binary system, we need to include the binary orbital parameters in the definition of the signal.

- The signal (1.1) is written in the preferred polarization basis $\overleftrightarrow{e}_{+,\times}$ constructed from unit vectors $\vec{\ell}$ and $\vec{m}$ orthogonal to each other and the propagation direction $\vec{k}$, with $\vec{\ell}$ lying in the neutron star's equatorial plane. This means they depend on the orientation of the neutron star's rotation axis, which is a parameter of the system. We can relate them to the canonical basis $\overleftrightarrow{\varepsilon}_{+,\times}$ constructed out of the vectors $\vec{\imath}$ and $\vec{\jmath}$ pointing West and North on the sky:

$$\overleftrightarrow{e}_{+} = \overleftrightarrow{\varepsilon}_{+} \cos 2\psi + \overleftrightarrow{\varepsilon}_{\times} \sin 2\psi \tag{1.7a}$$
$$\overleftrightarrow{e}_{\times} = -\overleftrightarrow{\varepsilon}_{+} \sin 2\psi + \overleftrightarrow{\varepsilon}_{\times} \cos 2\psi \tag{1.7b}$$

1.2 Exercise: JKS decomposition

By using (1.7) and the angle sum formulas, expand out (1.1) so that its ϕ_0 and ψ dependence is explicit. Show that it can be written

$$\overleftrightarrow{h}(\tau) = \mathcal{A}^{\mu} \overleftrightarrow{h}_{\mu}(\tau) \tag{1.8}$$

(using the Einstein summation convention that repeated upper and lower indices are summed over, now with $\sum_{\mu=1}^{4}$) where

$$\mathcal{A}^1 = A_+ \cos 2\psi \cos \phi_0 - A_\times \sin 2\psi \sin \phi_0 \tag{1.9a}$$
$$\mathcal{A}^2 = A_+ \sin 2\psi \cos \phi_0 + A_\times \cos 2\psi \sin \phi_0 \tag{1.9b}$$
$$\mathcal{A}^3 = -A_+ \cos 2\psi \sin \phi_0 - A_\times \sin 2\psi \cos \phi_0 \tag{1.9c}$$
$$\mathcal{A}^4 = -A_+ \sin 2\psi \sin \phi_0 + A_\times \cos 2\psi \cos \phi_0 \ . \tag{1.9d}$$

and

$$\overleftrightarrow{h}_1(\tau) = \overleftrightarrow{\varepsilon}_{+} \cos \Phi(\tau) \tag{1.10a}$$
$$\overleftrightarrow{h}_2(\tau) = \overleftrightarrow{\varepsilon}_{\times} \cos \Phi(\tau) \tag{1.10b}$$
$$\overleftrightarrow{h}_3(\tau) = \overleftrightarrow{\varepsilon}_{+} \sin \Phi(\tau) \tag{1.10c}$$
$$\overleftrightarrow{h}_4(\tau) = \overleftrightarrow{\varepsilon}_{\times} \sin \Phi(\tau) \tag{1.10d}$$

2 Data Analysis Method

2.1 Likelihood function

The combinations (1.9), first discovered by Jaranowski, Królak and Schutz[1] mean that the parameters h_0, ι, ψ and ϕ_0 can be treated

[1] P. Jaranowski, A. Królak, and B. F. Schutz, *Phys Rev D* **58**, 062001 (1998), hereafter JKS. Note that the decomposition is not unique. J. T. Whelan, R. Prix, C. J. Cutler, and J. L. Willis, arXiv:1311.0065 [to appear in *CQG*] exhibit an alternative decomposition which is more closely connected to the physical amplitude parameters.

differently from the other signal parameters. They're known as *amplitude parameters*, as compared with the phase parameters (sky position, frequency, spindowns, binary orbital parameters, etc), also known as Doppler parameters, because most of them are associated with the Doppler modulation of the signal. We refer to them collectively as λ.

The signal in a detector at time t is then

$$h(t) = \overleftrightarrow{h}(\tau(t)) : \overleftrightarrow{d} = \mathcal{A}^\mu h_\mu(t;\lambda) \tag{2.1}$$

where the four signal waveforms are

$$h_1(t;\lambda) = a(t;\lambda)\, \cos\Phi(t;\lambda) \tag{2.2a}$$
$$h_2(t;\lambda) = b(t;\lambda)\, \cos\Phi(t;\lambda) \tag{2.2b}$$
$$h_3(t;\lambda) = a(t;\lambda)\, \sin\Phi(t;\lambda) \tag{2.2c}$$
$$h_4(t;\lambda) = b(t;\lambda)\, \sin\Phi(t;\lambda) \tag{2.2d}$$

where $a = \overleftrightarrow{\varepsilon}_+ : \overleftrightarrow{d}$ and $b = \overleftrightarrow{\varepsilon}_\times : \overleftrightarrow{d}$ are the amplitude modulation coëfficients for that detector and sky position. They also change slowly with time, since the rotation of the Earth changes the geometry of the double-dot products $\overleftrightarrow{\varepsilon}_{+,\times} : \overleftrightarrow{d}$. (The detector tensor $\overleftrightarrow{d}$ has constant components in a basis co-rotating with the Earth, and the polarization basis tensors $\overleftrightarrow{\varepsilon}_{+,\times}$ have constant components in an inertial basis.)

If we divide time up into shorter intervals of duration T, things are a little easier to describe; within a short timeframe, the signals are approximately monochromatic, and the AM coëfficients are approximately constant. Notationally, we add a label X to refer to the detector and I to refer to the time interval. If the data in instrument X during time interval I is

$$x^X_{Ij} = x^X(t_{I0} + j\delta t) \tag{2.3}$$

then we write its Fourier transform (also known as a "short Fourier transform" or SFT) as

$$\sum_{j=0}^{N-1} \delta t\, x^X_{Ij}\, e^{-i2\pi jk/N} \sim \int_{t_{I0}}^{t_{I0}+T} dt\, x(t)\, e^{-i2\pi f_k(t-t_{I0})} \equiv \widetilde{x}^X_I(f_k) \tag{2.4}$$

where $T = N\,\delta t$, $f_k = k\,\delta f$, $\delta f = \frac{N}{\delta t}$, etc. The signal contribution to the Ith SFT in detector X is

$$\widetilde{h}^X_I(f) = \mathcal{A}^\mu \widetilde{h}^X_{I,\mu}(f;\lambda) \tag{2.5}$$

where

$$\widetilde{h}^X_{I,1}(f;\lambda) = a^X_I(\lambda)\, \widetilde{\cos\Phi^X_I}(f;\lambda) \tag{2.6a}$$
$$\widetilde{h}^X_{I,2}(f;\lambda) = b^X_I(\lambda)\, \widetilde{\cos\Phi^X_I}(f;\lambda) \tag{2.6b}$$
$$\widetilde{h}^X_{I,3}(f;\lambda) = a^X_I(\lambda)\, \widetilde{\sin\Phi^X_I}(f;\lambda) \tag{2.6c}$$
$$\widetilde{h}^X_{I,4}(f;\lambda) = b^X_I(\lambda)\, \widetilde{\sin\Phi^X_I}(f;\lambda) \tag{2.6d}$$

If the data in each detector consist of Gaussian noise $n^X_I(t)$ with a one-sided power spectral density $S^X_I(f)$, plus the signal,

$$\widetilde{x}^X_I(f) = \widetilde{n}^X_I(f) + \mathcal{A}^\mu \widetilde{h}^X_{I,\mu}(f;\lambda) \tag{2.7}$$

then the probability density of the data can be written as

$$P(x|\mathcal{A},\lambda) \propto e^{\Lambda(x|\mathcal{A},\lambda)} \tag{2.8}$$

where the log-likelihood is

$$\Lambda(x|\mathcal{A},\lambda) = -2\sum_I\sum_X \int_0^{f_{\mathrm{Ny}}} df\, \frac{\left|\widetilde{n}^X_I(f)\right|^2}{S^X_I(f)} = -\frac{1}{2}\langle x-h|x-h\rangle \tag{2.9}$$

and we've defined the inner product

$$\langle y|z\rangle = 4\,\mathrm{Re}\sum_I\sum_X \int_0^{f_{\mathrm{Ny}}} df\, \frac{\widetilde{y}^X_I(f)^*\, \widetilde{z}^X_I(f)}{S^X_I(f)} \tag{2.10}$$

On the other hand, if the data consist only of noise, then the probability density is

$$P(x|\mathcal{H}_N) \propto e^{-\frac{1}{2}\langle x|x\rangle} \tag{2.11}$$

That makes the log-likelihood ratio

$$\begin{aligned}\ln \frac{P(x|\mathcal{A},\lambda)}{P(x|\mathcal{H}_N)} &= \frac{1}{2}\left[\langle x|h(\mathcal{A},\lambda)\rangle + \langle h(\mathcal{A},\lambda)|x\rangle - \langle h(\mathcal{A},\lambda)|h(\mathcal{A},\lambda)\rangle\right] \\ &= \mathcal{A}^\mu \langle h_\mu(\lambda)|x\rangle - \frac{1}{2}\mathcal{A}^\mu \langle h_\mu(\lambda)|h_\nu(\lambda)\rangle \mathcal{A}^\mu \\ &= \mathcal{A}^\mu x_\mu(\lambda) - \frac{1}{2}\mathcal{A}^\mu \mathcal{M}_{\mu\nu}(\lambda)\mathcal{A}^\nu\end{aligned} \tag{2.12}$$

in terms of a "metric"

$$\mathcal{M}_{\mu\nu}(\lambda) = \langle h_\mu(\lambda)|h_\nu(\lambda)\rangle = 4\,\mathrm{Re}\sum_I \sum_X \int_0^{f_{\mathrm{Ny}}} df\, \frac{\widetilde{h}^X_{\mu I}(f;\lambda)^* \,\widetilde{h}^X_{\nu I}(f;\lambda)}{S^X_I(f)} \tag{2.13}$$

on amplitude parameter space and a data-vector

$$x_\mu(\lambda) = \langle h_\mu(\lambda)|x\rangle = 4\,\mathrm{Re}\sum_I \sum_X \int_0^{f_{\mathrm{Ny}}} df\, \frac{\widetilde{h}^X_{\mu I}(f;\lambda)^* \,\widetilde{x}^X_I(f)}{S^X_I(f)} \tag{2.14}$$

2.2 Detection Statistic

To search for gravitational waves, we need to have a way to decide between a signal hypothesis $\mathcal{H}_S$ and the noise hypothesis $\mathcal{H}_N$. The Neyman-Pearson lemma says that the best detection statistic we can use is a monotonic function of the likelihood ratio

$$\frac{P(x|\mathcal{H}_S)}{P(x|\mathcal{H}_N)} \tag{2.15}$$

In general, the signal hypothesis $\mathcal{H}_S$ represents a family of signals with different values for the signal parameters λ and $\mathcal{A}$, and the likelihood ratio involves marginalizing over those parameters:

$$\int d\mathcal{A}\, d\lambda\, P(\mathcal{A},\lambda|\mathcal{H}_S)\, \frac{P(x|\mathcal{A},\lambda)}{P(x|\mathcal{H}_N)} \tag{2.16}$$

This has been considered recently[2], but choosing a physical prior on the signal parameters leads to a more mathematically involved integration problem. So we'll focus for now on a method developed by JKS, where the likelihood ratio is analytically maximized over the amplitude parameters to produce a maximum-likelihood statistic

$$\mathcal{F}(\lambda) = \max_{\mathcal{A}} \ln \frac{P(x|\mathcal{A},\lambda)}{P(x|\mathcal{H}_N)} = \frac{1}{2}\widehat{\mathcal{A}}^\mu \mathcal{M}_{\mu\nu}\widehat{\mathcal{A}}^\nu \tag{2.17}$$

where, if we define $\mathcal{M}^{\mu\nu}$ as the matrix inverse of $\mathcal{M}_{\mu\nu}$, the maximum-likelihood values for the amplitude parameters are

$$\widehat{\mathcal{A}}^\mu = \mathcal{M}^{\mu\nu} x_\nu \tag{2.18}$$

and the $\mathcal{F}$ statistic is

$$\mathcal{F}(\lambda) = \frac{1}{2} x_\mu(\lambda)\mathcal{M}^{\mu\nu}(\lambda) x_\nu(\lambda) \tag{2.19}$$

Note that even constructing this statistic means choosing values for the phase parameters λ: sky position, frequency, spindowns, etc. In the case of a targeted search for gravitational waves from a neutron star seen as a radio or X-ray pulsar, this is no big deal; we know the sky position and the spin, including its evolution. When looking for unknown objects, we have to try a bunch of different points in parameter space. How far off you can be in parameter space before the detection statistic becomes ineffective

[2] R. Prix and B. Krishnan, *CQG* **26**, 204013 (2009)

depends on the coherent observing time: how far apart the earliest and latest observations are. For a fully coherent search, the number of points needed to cover parameter space grows rapidly with observing time, until the coherent search becomes impossible. Instead so-called semi-coherent methods, which are the topic of a different lecture, must be used.

2.3 Statistical properties of the $\mathcal{F}$-statistic

Focus now on the case where the phase parameters are known. If the unknown amplitude parameters are $\{\mathcal{A}^\mu\}$, we can show that the data vector has expectation value

$$E\left[x_\mu\right] = \mathcal{M}_{\mu\nu}\mathcal{A}^\nu = \mu_\mu \tag{2.20}$$

and variance

$$E\left[(x_\mu - \mu_\mu)(x_\nu - \mu_\nu)\right] = \mathcal{M}_{\mu\nu} \tag{2.21}$$

Since it's a linear combination of Gaussian data, it is itself Gaussian, i.e., its pdf is

$$P(\{x_\mu\}|\mathcal{A}) = \left[\det(2\pi\mathcal{M}_{\mu\nu})\right]^{-1/2} \exp\left(\frac{1}{2}(x_\mu - \mu_\mu)\mathcal{M}^{\mu\nu}(x_\nu - \mu_\nu)\right) \tag{2.22}$$

or equivalently, the pdf of the maximum likelihood estimates $\{\widehat{\mathcal{A}}^\mu\}$ is

$$P(\{\widehat{\mathcal{A}}^\mu\}|\mathcal{A}) = \left[\det(2\pi\mathcal{M}^{\mu\nu})\right]^{-1/2} \exp\left(\frac{1}{2}(\widehat{\mathcal{A}}^\mu - \mathcal{A}^\mu)\mathcal{M}_{\mu\nu}(\widehat{\mathcal{A}}^\nu - \mathcal{A}^\nu)\right) \tag{2.23}$$

$$2\mathcal{F} = x_\mu\mathcal{M}^{\mu\nu}x_\nu = \widehat{\mathcal{A}}^\mu\mathcal{M}_{\mu\nu}\widehat{\mathcal{A}}^\nu \tag{2.24}$$

obeys what's known as a non-central chi-squared distribution with four degrees of freedom and non-centrality parameter $\mathcal{A}^\mu\mathcal{M}_{\mu\nu}\mathcal{A}^\nu$. This means

$$E\left[2\mathcal{F}\right] = 4 + \mathcal{A}^\mu\mathcal{M}_{\mu\nu}\mathcal{A}^\nu \tag{2.25}$$