Treatment of Calibration Uncertainty in Multi-Baseline Cross-Correlation Searches for Gravitational Waves

John T. Whelan

Center for Computational Relativity and Gravitation and School of Mathematical Sciences, Rochester Institute of Technology, 85 Lomb Memorial Drive, Rochester NY 14623, USA

E-mail: `john.whelan@astro.rit.edu`

Emma L. Robinson

Max-Planck-Institut für Gravitationsphysik (Albert-Einstein-Institut), D-14476 Potsdam, Germany

Joseph D. Romano

Center for Gravitational-Wave Astronomy and Department of Physics and Astronomy, The University of Texas at Brownsville, 80 Fort Brown, Brownsville TX 78520, USA

Eric H. Thrane

Department of Physics, University of Minnesota, Minneapolis, MN, USA

Abstract. Uncertainty in the calibration of gravitational-wave (GW) detector data leads to systematic errors which must be accounted for in setting limits on the strength of GW signals. When cross-correlation measurements are made using data from a pair of instruments, as in searches for a stochastic GW background, the calibration uncertainties of the individual instruments can be combined into an uncertainty associated with the pair. With the advent of multi-baseline GW observation (e.g., networks consisting of multiple detectors such as the LIGO observatories and Virgo), a more sophisticated treatment is called for. We describe how the correlations between calibration factors associated with different pairs can be taken into account by marginalizing over the uncertainty associated with each instrument.

1. Calibration Uncertainty with One Baseline

Consider an experiment to measure a physical quantity μ (e.g., the stochastic GW background strength $\Omega_{\text{gw}}(f)$ [1, 2]). An optimal combination x of cross-correlation measurements provides a point estimate of μ with error bar σ. Given a likelihood function $p(x|\mu)$ and a prior $p(\mu)$, one can use Bayes's Theorem to construct the posterior

$$p(\mu|x) = \frac{p(x|\mu)p(\mu)}{p(x)} = \frac{p(x|\mu)p(\mu)}{\int d\mu \, p(x|\mu) \, p(\mu)} \,. \tag{1}$$

Due to calibration uncertainties in each of the instruments that make up the baseline for the cross-correlation, x is an estimator not of μ, but of $\lambda\mu$, where λ is an unknown calibration factor (for the baseline) described by an uncertainty ε. Thus the likelihood function also depends on the calibration factor λ

$$p(x|\mu, \lambda) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - \lambda\mu)^2}{2\sigma^2}\right] , \tag{2}$$

so the posterior (1) is now constructed from the *marginalized* likelihood

$$p(x|\mu) = \int d\lambda \, p(x|\mu, \lambda)p(\lambda) . \tag{3}$$

If we assume a Gaussian distribution for λ,

$$p(\lambda) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left[-\frac{(\lambda - 1)^2}{2\varepsilon^2}\right] , \tag{4}$$

then we can do the marginalization analytically if the range of λ values is taken to be $(-\infty, \infty)$.[1] This leads to

$$p(x|\mu) = \frac{1}{\sqrt{2\pi(\sigma^2 + \varepsilon^2\mu^2)}} \exp\left[-\frac{1}{2}\frac{(x - \mu)^2}{(\sigma^2 + \varepsilon^2\mu^2)}\right] . \tag{5}$$

This is the method used in stochastic GW searches with two LIGO sites, e.g., [3, 4].

A more physically-motivated prior, which explicitly takes into account that λ is multiplicative and takes on only positive values, is a log-normal distribution

$$p(\lambda) = \frac{1}{\lambda\varepsilon\sqrt{2\pi}} \exp\left[-\frac{(\ln\lambda)^2}{2\varepsilon^2}\right] \quad \text{or} \quad p(\Lambda) = \frac{1}{\varepsilon\sqrt{2\pi}} \exp\left[-\frac{\Lambda^2}{2\varepsilon^2}\right] \quad \text{where } \Lambda = \ln\lambda . \tag{6}$$

This was the approach taken in the stochastic GW search using LIGO and ALLEGRO [5], but has the drawback of requiring numerical integration over Λ because

$$p(x|\mu, \Lambda) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{(x - e^{\Lambda}\mu)^2}{2\sigma^2}\right] \tag{7}$$

gives a factor which is not Gaussian in Λ. Figure 1 compares the posterior distributions for the two different choices of priors for different measured values of x and different values of ε.

2. Calibration Uncertainty with Multiple Baselines

With more than two instruments, there are multiple baselines and multiple calibration uncertainties to marginalize over. For instance, the stochastic background search using initial LIGO and Virgo data [6, 7] involved 4 different instruments $I \in \{H1, H2, L1, V1\}$ and 5 different baselines $\alpha \in \{H1L1, H1V1, H2L1, H2V1, L1V1\}$.

Since the cross-correlation measurements for different baselines involve different calibration factors, all of the baselines cannot be optimally combined before marginalizing over calibration. Instead, all of the measurements for a baseline α can be combined into a single point estimate x_α with error bar σ_α. Each baseline has unknown calibration factor λ_α. Since the statistical errors for the different baselines are independent [8], the likelihood is the product

$$p(\mathbf{x}|\mu, \boldsymbol{\lambda}) = \prod_\alpha \left\{ \frac{1}{\sigma_\alpha\sqrt{2\pi}} \exp\left[-\frac{(x_\alpha - \lambda_\alpha\mu)^2}{2\sigma_\alpha^2}\right] \right\} , \tag{8}$$

---

[1] Since λ is an amplitude calibration factor, it should take on only positive values. But for small ε—e.g., of order 0.10 to 0.20—integrating over λ from 0 to ∞ is numerically the same as integrating over λ from $-\infty$ to $+\infty$.
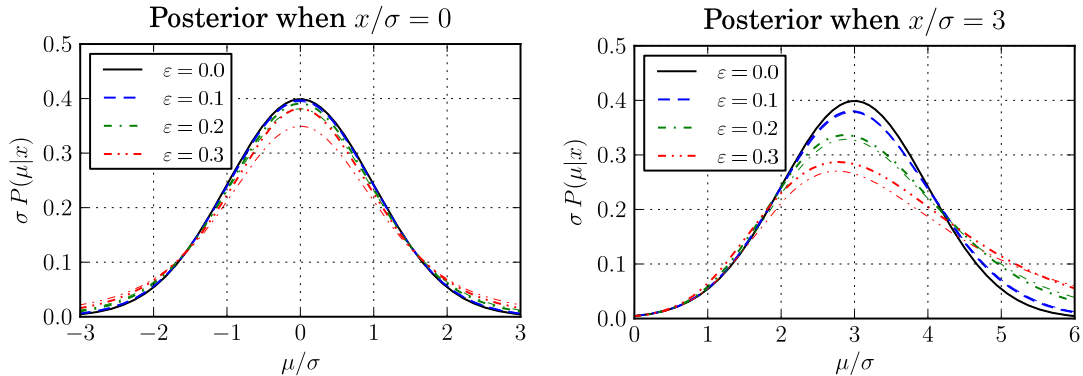
Figure 1. Effects of marginalizing analytically over a single calibration factor for two different values of the measured cross-correlation, $x = 0$ and $x = 3\sigma$. The thick line is a numerical marginalization with a log-normal prior on λ; the thin line is an analytic marginalization with a Gaussian prior on λ.

where $\mathbf{x} \equiv \{x_\alpha\}$ and $\boldsymbol{\lambda} \equiv \{\lambda_\alpha\}$. The calibration factor λ_α for each baseline is $\lambda_{IJ} = \xi_I \xi_J$, and is determined by the per-instrument calibration factors $\boldsymbol{\xi} \equiv \{\xi_I\}$. If each instrument's calibration has an underlying uncertainty δ_I, the per-baseline calibration factors $\boldsymbol{\lambda}$ have the following means, variances and covariances:

$$\langle \lambda_{IJ} \rangle = 1\,, \quad \langle \lambda_{IJ}\lambda_{IJ} \rangle = 1 + \delta_I{}^2 + \delta_J{}^2 + \mathcal{O}(\delta^4); \quad \langle \lambda_{IJ}\lambda_{JK} \rangle = 1 + \delta_J{}^2 + \mathcal{O}(\delta^4) \quad \text{if } I \neq K \,. \quad (9)$$

2.1. Per-Baseline Calibration Marginalization

One approach is to marginalize over the per-baseline calibration factors assuming a multivariate Gaussian prior $p(\boldsymbol{\lambda})$. This has the advantage that the marginalization integral

$$p(\mathbf{x}|\mu) = \int d\boldsymbol{\lambda}\, p(\mathbf{x}|\mu, \boldsymbol{\lambda})\, p(\boldsymbol{\lambda}) \qquad (10)$$

can be done analytically if the integrals over the per-baseline calibration factors $\boldsymbol{\lambda}$ are taken over $(-\infty, \infty)$. However, the relationship $\lambda_{IJ} = \xi_I \xi_J$ implies that

$$\lambda_{IJ}\lambda_{KL} - \lambda_{IK}\lambda_{JL} = 0 \,. \qquad (11)$$

For a multivariate Gaussian prior on $\boldsymbol{\lambda}$, this relation is true *only as an expectation value*, not as an identity for all values of λ_{IJ}, λ_{KL}, etc.

2.2. Per-Instrument Calibration Marginalization

An alternative approach which enforces identities such as (11) is to set a prior which is the product of independent priors on each per-instrument calibration factor ξ_I or equivalently on $\Xi_I = \ln \xi_I$. Similarly defining the log-calibration factor for a baseline $\Lambda_\alpha = \ln \lambda_\alpha$, we have

$$\Lambda_{IJ} = \ln \lambda_{IJ} = \ln(\xi_I \xi_J) = \Xi_I + \Xi_J \,. \qquad (12)$$

The likelihood is

$$p(\mathbf{x}|\mu, \boldsymbol{\Xi}) = \prod_\alpha \left\{ \frac{1}{\sigma_\alpha \sqrt{2\pi}} \exp\left[-\frac{(x_\alpha - e^{\Lambda_\alpha}\mu)^2}{2\sigma_\alpha{}^2} \right] \right\} \qquad (13)$$

and the marginalized likelihood is

$$p(\mathbf{x}|\mu) = \int d\mathbf{\Xi}\, p(\mathbf{x}|\mu, \mathbf{\Xi})\, p(\mathbf{\Xi})\ . \tag{14}$$

An obvious prior is log-normal on ξ_I, i.e., Gaussian on Ξ_I:

$$p(\mathbf{\Xi}) = \prod_I \left\{ \frac{1}{\delta_I \sqrt{2\pi}}\, \exp\left(\frac{\Xi_I{}^2}{2\delta_I{}^2}\right) \right\}\ . \tag{15}$$

The exact integral over $\mathbf{\Xi}$ would need to be done numerically for each μ, but if $\boldsymbol{\delta}$ are small, one can make the approximation $e^{\Lambda_{IJ}} \approx 1 + \Lambda_{IJ} = 1 + \Xi_I + \Xi_J$ to convert the likelihood to a Gaussian integral over $\mathbf{\Xi}$ which can be done analytically. The result is a likelihood of the form

$$p(\mathbf{x}|\mu) = \sqrt{\det\left(\frac{\mathbf{M}(\mu, \boldsymbol{\sigma}, \boldsymbol{\delta})}{2\pi}\right)} \exp\left[-\frac{1}{2}\sum_\alpha \sum_\beta (x_\alpha - \mu)\, M_{\alpha\beta}(\mu, \boldsymbol{\sigma}, \boldsymbol{\delta})\,(x_\beta - \mu)\right]\ . \tag{16}$$

This is the approach which was used for the multi-baseline upper limits in [7].

For the special case of two instruments which make up a single baseline, the matrix $\mathbf{M}$ reduces to a single number with value

$$\mathbf{M}(\mu, \boldsymbol{\sigma}, \boldsymbol{\delta}) \equiv \frac{1}{\sigma_{12}^2 + \mu^2(\delta_1{}^2 + \delta_2{}^2)}\ . \tag{17}$$

Comparing (17) to (5), we see that this approximation gives the same result as assuming a Gaussian prior in λ_{12} with $\varepsilon_{12}{}^2 = \delta_1{}^2 + \delta_2{}^2$.

3. Ongoing Work

More generally, we may be using $\mathbf{x}$ to estimate multiple physical quantities, such as different spherical harmonic modes of a non-isotropic stochastic GW background [9, 10]. These methods of analytic marginalization with either a multivariate Gaussian prior or an approximate likelihood function can be applied to the effects of calibration uncertainty in that search as well. Additionally, these calibration effects may also be considered in other cross-correlation searches, such as the modelled cross-correlation search for periodic GW signals.[11]

References
[1] Christensen N L 1992 *Phys. Rev.* **D46**, 5250
[2] Allen B and Romano JD 1999 *Phys. Rev.* **D59**, 102001
[3] Abbott B et al (LIGO Scientific Collaboration) 2007 *Astrophys. J* **659**, 918
[4] Abbott B P et al (LIGO Scientific Collaboration and Virgo) 2009 *Nature* **460**, 990
[5] Abbott B et al (LIGO Scientific Collaboration and ALLEGRO) 2007 *Phys. Rev.* **D76**, 022001
[6] Cella G et al 2007 *Class. Quant. Grav.* **24**, S639
[7] Abadie J et al (LIGO Scientific Collaboration and Virgo) 2012 *Phys. Rev.* **D85**, 122001
[8] Lazzarini A et al 2004 *Phys. Rev.* **D70**, 062001
[9] Thrane E et al 2009 *Phys. Rev.* **D80**, 122002
[10] Abadie J et al (LIGO Scientific Collaboration and Virgo) 2011 *Phys. Rev. Lett.* **107**, 271102
[11] Dhurandhar S et al 2008 *Phys. Rev.* **D77**, 082001