Technical Note
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Subject: Analysis of LIGO Flexure Rods

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1. Introduction

This memorandum documents the stiffness and stress characteristics of flexure rods as used in the advanced LIGO seismic isolation concept. The analysis has been performed parametrically and in closed form, to allow initial design trades to take place. Final design verification will be completed by detailed finite element analysis, including the effects of nonrigid ends, fillets, and other details.

The advanced LIGO seismic isolation system utilizes three flexure rods per stage to provide soft lateral springs. (Vertical flexibility is controlled by separate leaf springs.) The flexure rods are nominally aligned with the gravity vector, and carry the weight of the suspended components. While carrying this axial load, the rods are subjected to small lateral loads. The rod ends are rotationally fixed.

Figure 1a shows this loading scenario. The axial load P is the weight carried by the rod. Lateral loads V_1 and V_2 are applied at the ends of the rod. Moments M_1 and M_2 are also applied at the ends.

The design process will require selection of the rod length and cross sectional area to give the desired lateral stiffness, while keeping stress levels sufficiently low.

2. Beam Equation and Solution

2.1. Assumptions

For design purposes, we will assume the beam slenderness ratio is such that it can be treated as an Euler bending beam (ignoring transverse shear effects). Deviations from this assumption will be quantified during design validation.

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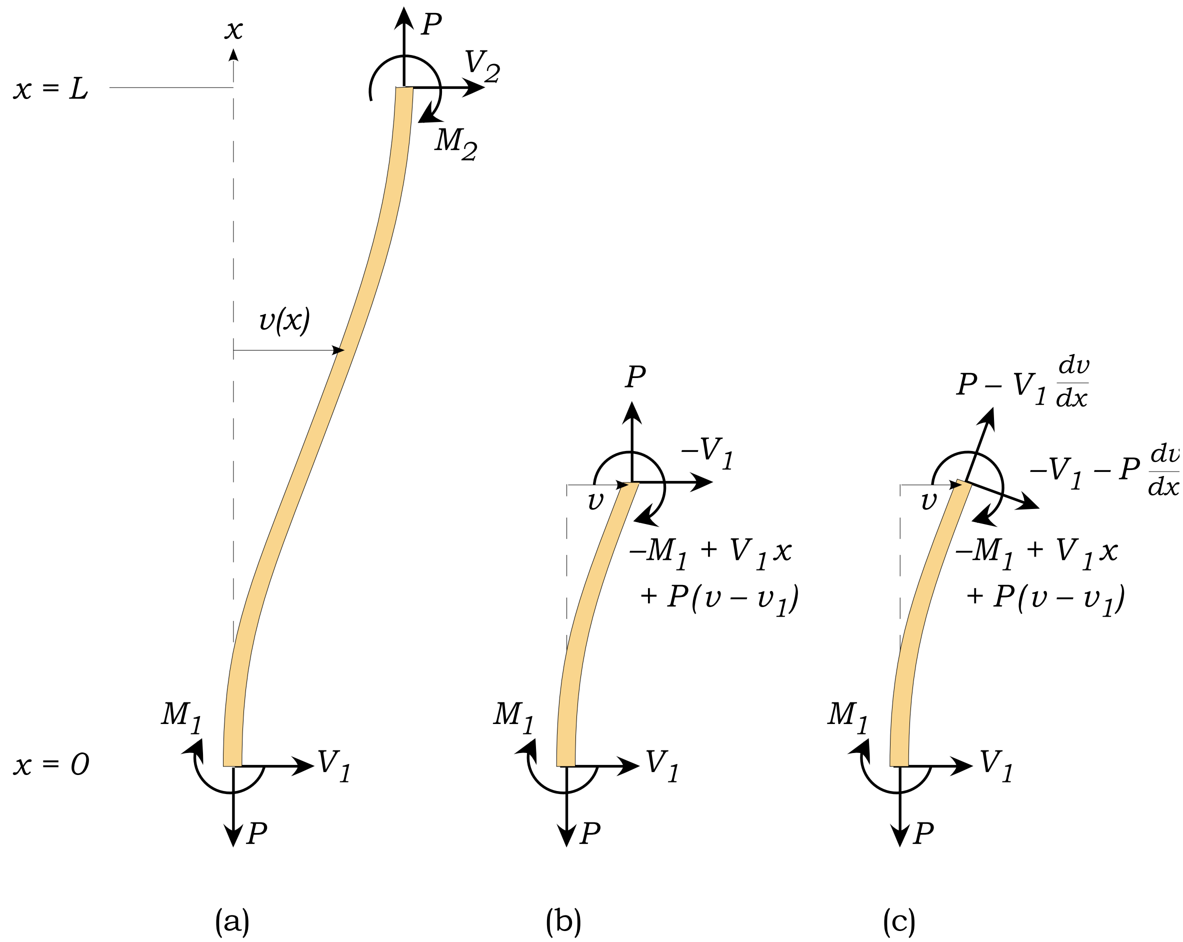

Figure 1. Free body diagram of slender beam under combined axial and lateral loading: (a) Full beam with balanced end loads and moments; (b) Section cut through intermediate station; (c) Section cut through intermediate station perpendicular to beam axis.

2.2. Beam Equation

As illustrated in Figure 1, the displacement of the beam centerline is $v(x)$, where x is the distance from one end of the beam. The displacement at $x = 0$ is v_1. Considering Figure 1b, force balance requires constant shear along the beam (in a direction parallel to the end planes), but the moment along the beam is modified by the term $P(v - v_1)$, in addition to the linear variation to balance the end shear. The shear and moment can be resolved in the beam local coordinate system as shown in Figure 1c. Note that dv/dx is assumed small.

Let the cross sectional area of the rod be A, its area moment of inertia be I, and its Young's modulus be E. Equating the moment in the beam to $EI\, d^2v/dx^2$, we have

$$EI \frac{d^2v}{dx^2} = -M_1 + V_1 x + P(v - v_1). \tag{1}$$

Differentiating this equation twice gives the standard equation for a bending beam under axial load:

$$\textit{Beam equation:} \qquad \boxed{\frac{d^2}{dx^2}\left(EI\frac{d^2v}{dx^2}\right) - P\frac{d^2v}{dx^2} = 0} \tag{2}$$

For a constant cross section with $P > 0$, equation (2) admits solutions of the form

$$v(x) = c_1 + c_2 x + c_3 \cosh Kx + c_4 \sinh Kx, \tag{3}$$

where

$$K = \sqrt{\frac{P}{EI}}\ . \tag{4}$$

2.3. Deflected Shape

The constants in equation (3) can be determined to match the displacements and rotations of the two ends of the rod:

$$v(0) = v_1\ , \qquad \left.\frac{dv}{dx}\right|_{x=0} = \theta_1\ , \qquad v(L) = v_2\ , \qquad \left.\frac{dv}{dx}\right|_{x=L} = \theta_2. \tag{5}$$

The solution can be written in the form

$$\begin{aligned} v(x) = \\ & v_1 \cdot \frac{1}{2}\left[1 - \frac{K\left(x-\frac{L}{2}\right)\cosh\frac{KL}{2} - \sinh K\left(x-\frac{L}{2}\right)}{\frac{KL}{2}\cosh\frac{KL}{2} - \sinh\frac{KL}{2}}\right] \\ & + v_2 \cdot \frac{1}{2}\left[1 + \frac{K\left(x-\frac{L}{2}\right)\cosh\frac{KL}{2} - \sinh K\left(x-\frac{L}{2}\right)}{\frac{KL}{2}\cosh\frac{KL}{2} - \sinh\frac{KL}{2}}\right] \\ & + \theta_1 \cdot \frac{1}{2K}\left[\frac{\cosh\frac{KL}{2} - \cosh K\left(x-\frac{L}{2}\right)}{\sinh\frac{KL}{2}} - \frac{K\left(x-\frac{L}{2}\right)\sinh\frac{KL}{2} - \frac{KL}{2}\sinh K\left(x-\frac{L}{2}\right)}{\frac{KL}{2}\cosh\frac{KL}{2} - \sinh\frac{KL}{2}}\right] \\ & + \theta_2 \cdot \frac{1}{2K}\left[-\frac{\cosh\frac{KL}{2} - \cosh K\left(x-\frac{L}{2}\right)}{\sinh\frac{KL}{2}} - \frac{K\left(x-\frac{L}{2}\right)\sinh\frac{KL}{2} - \frac{KL}{2}\sinh K\left(x-\frac{L}{2}\right)}{\frac{KL}{2}\cosh\frac{KL}{2} - \sinh\frac{KL}{2}}\right]. \end{aligned} \tag{6}$$

The deflected shape of the beam is plotted in Figure 2 for selected values of the dimensionless parameter KL, for the special case of clamped ends ($\theta_1 = \theta_2 = 0$) and with $v_1 = 0$. Note that as KL increases, the beam shape changes from a cubic to more of a straight line.

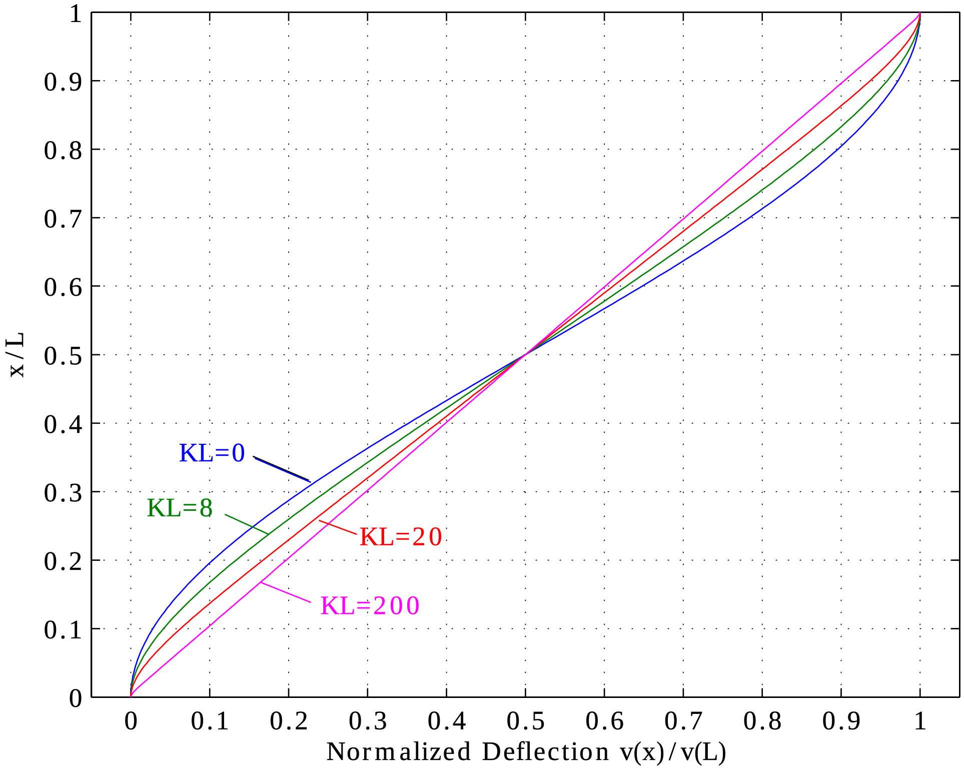

Figure 2. Beam deflection shape for various values of KL, with $v_1 = \theta_1 = \theta_2 = 0$.

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3. Stiffness Matrix

3.1. End Forces and Moments

To develop a stiffness matrix, we require the end forces and moments in terms of the end displacements and rotations. Referring to equation (1) (and its derivative), we find that

$$V_1 = \frac{d}{dx}\Big(EI\frac{d^2v}{dx^2} - Pv\Big)\Big|_{x=0} \ , \tag{7}$$

$$M_1 = -\, EI\frac{d^2v}{dx^2}\Big|_{x=0} \ . \tag{8}$$

Force and moment balance (see Figure 1b) requires that

$$V_2 = -V_1 \ , \tag{9}$$

$$M_2 = -M_1 + V_1 L + P(v_2 - v_1). \tag{10}$$

3.2. Matrix Formulation

Substituting $v(x)$ from equation (6) into equations (7) through (10), and writing the result in matrix form, we obtain

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \begin{bmatrix} \kappa_a & \kappa_b & -\kappa_a & \kappa_b \\ \kappa_b & \kappa_c & -\kappa_b & \kappa_d \\ -\kappa_a & -\kappa_b & \kappa_a & -\kappa_b \\ \kappa_b & \kappa_d & -\kappa_b & \kappa_c \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}, \tag{11}$$

where

$$\kappa_a = \frac{PK}{2} \frac{1}{\frac{KL}{2} - \tanh\frac{KL}{2}}, \tag{12}$$

$$\kappa_b = \frac{P}{2} \frac{\tanh\frac{KL}{2}}{\frac{KL}{2} - \tanh\frac{KL}{2}}, \tag{13}$$

$$\kappa_c = \frac{P}{2K} \left(\frac{\frac{KL}{2}\tanh\frac{KL}{2}}{\frac{KL}{2} - \tanh\frac{KL}{2}} + \coth\frac{KL}{2} \right), \tag{14}$$

$$\kappa_d = \frac{P}{2K} \left(\frac{\frac{KL}{2}\tanh\frac{KL}{2}}{\frac{KL}{2} - \tanh\frac{KL}{2}} - \coth\frac{KL}{2} \right). \tag{15}$$

3.3. Limiting Case

As a check, note that when the axial preload P is removed, the stiffness matrix of the beam should reduce to the well-known stiffness matrix of a standard Euler bending beam. We cannot simply substitute $P = 0$ into the matrix formulas, but rather must take the limit as $P \to 0$. For P (and therefore K) small, we find

$$\kappa_a = \frac{12EI}{L^3}(1 + \frac{K^2L^2}{10} + \cdots), \tag{16}$$

$$\kappa_b = \frac{6EI}{L^2}(1 + \frac{K^2L^2}{60} + \cdots), \tag{17}$$

$$\kappa_c = \frac{4EI}{L}(1 + \frac{K^2L^2}{30} + \cdots), \tag{18}$$

$$\kappa_d = \frac{2EI}{L}(1 - \frac{K^2L^2}{60} + \cdots). \tag{19}$$

Thus in the limit as $P \to 0$, the stiffness matrix becomes, as expected:

$$\begin{Bmatrix} V_1 \\ M_1 \\ V_2 \\ M_2 \end{Bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}. \tag{20}$$

3.4. Pendulum Interpretation

In the expected operation of the flexure rod, the ends are guided (zero slope). Referring to equation (11) with $\theta_1 = \theta_2 = 0$, we find that the end forces and moments are

$$V_1 = -V_2 = \kappa_a(v_1 - v_2) = \frac{PK}{2} \frac{1}{\frac{KL}{2} - \tanh\frac{KL}{2}}(v_1 - v_2) \,, \tag{21}$$

$$M_1 = M_2 = \kappa_b(v_1 - v_2) = \frac{PK}{2} \frac{\tanh\frac{KL}{2}}{\frac{KL}{2} - \tanh\frac{KL}{2}}(v_1 - v_2) \,. \tag{22}$$

Note that the combination of shear and moment at each end is equivalent to a pure shear load acting at a distance $M_1/V_1 = \kappa_b/\kappa_a$ away from the end, transmitted through rigid structure. This distance will be called the "zero moment distance" Z, and the corresponding pure shear locations will be called the "zero moment points".

Zero Moment Distance:
$$\boxed{Z = \kappa_b/\kappa_a = \frac{1}{K}\tanh\frac{KL}{2}} \tag{23}$$

The lateral stiffness k of the beam (with clamped ends) is simply κ_a. Using the definition of Z in equation (23), the expression for κ_a in equation (12) can be written as:

Lateral Stiffness:
$$\boxed{k = \frac{P}{L - 2Z}} \tag{24}$$

Equation (24) admits a simple intepretation. Recall that a simple pendulum of length ℓ has gravity-imparted lateral stiffness of P/ℓ. The lateral stiffness of the flexure rod is thus equal to that of a simple pendulum of length $L - 2Z$. In addition, while undergoing pure translation the rod reacts no moment to the upper side about a virtual point located at $x = L - Z$, and reacts no moment to the lower side through a virtual point located at $x = Z$. Thus with regard to lateral stiffness, the flexure rod is equivalent to a (pin-ended) pendulum connecting the virtual points $x = Z$ and $x = L - Z$.

The lateral stiffness can also be expressed in terms of natural frequency. Assume the upper end of the flexure rod is fixed in translation and rotation, and the lower end is rotationally fixed and is attached to a mass P/g. Then the natural frequency of the mass in lateral translation is simply

Natural Frequency:
$$\boxed{f_n = \frac{1}{2\pi}\sqrt{\frac{g}{L - 2Z}}} \tag{25}$$

as the pendulum analogy would suggest.

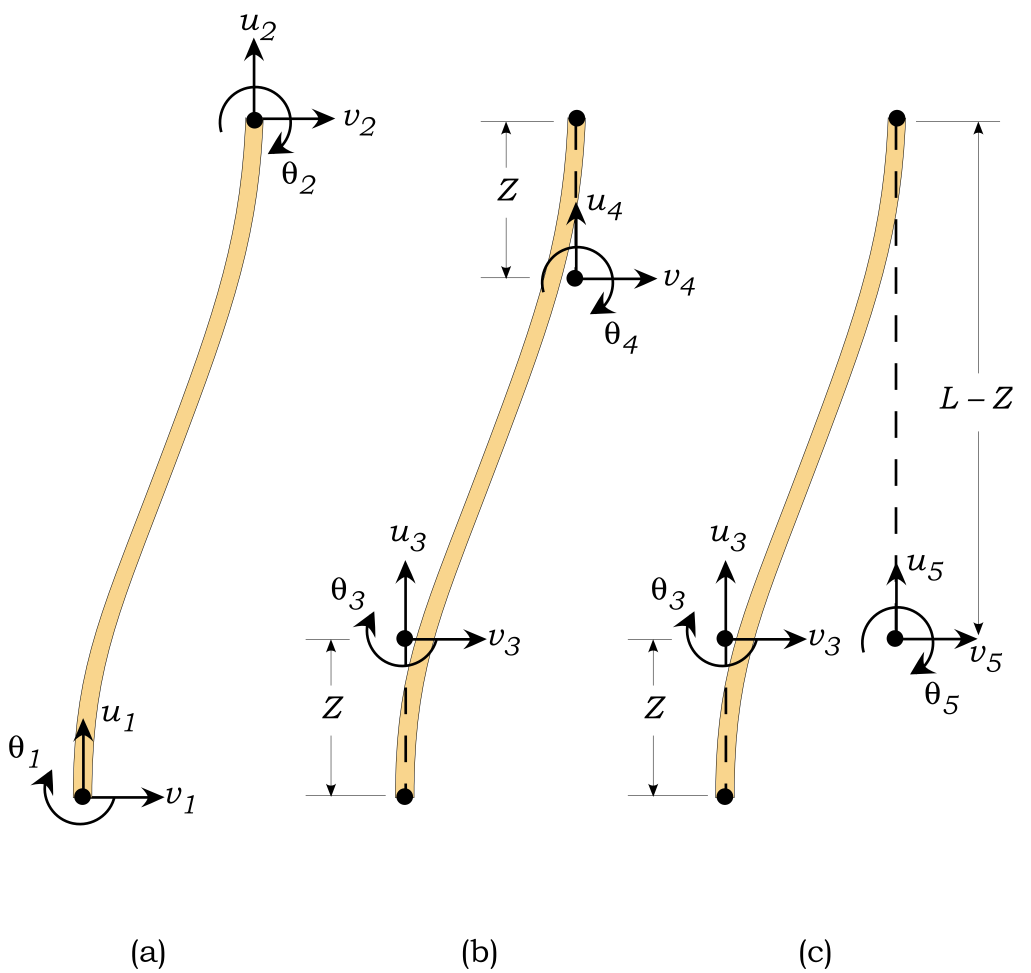

Figure 3. Various displacement coordinates used for the beam stiffness matrix: (a) Rod end displacements and rotations; (b) Displacements and rotations of lower zero moment point on suspended side, upper zero moment point on upper side; (c) Displacements and rotations of lower zero moment point on both sides.

3.5. Transformation to Zero Moment Points

The stiffness matrix simplifies considerably if we transform coordinates to the zero moment points. Following Figure 3b, make the change of variables

$$\begin{aligned} v_1 &= v_3 - Z\theta_3 & V_1 &= V_3 \\ \theta_1 &= \theta_3 & M_1 &= ZV_3 + M_3 \\ v_2 &= v_4 + Z\theta_4 & V_2 &= V_4 \\ \theta_4 &= \theta 3 & M_2 &= -ZV_4 + M_4 \end{aligned} \tag{26}$$

in the stifness equation (11). The 3 and 4 subscripts refer to displacements, rotations, forces, and moments at the lower and upper zero moment points respectively. After simplification, the transformed stiffness equation can be written

$$\begin{Bmatrix} V_3 \\ M_3 \\ V_4 \\ M_4 \end{Bmatrix} = \begin{bmatrix} \frac{P}{L-2Z} & 0 & -\frac{P}{L-2Z} & 0 \\ 0 & \frac{P}{2}\left(Z + \frac{1}{K^2 Z}\right) & 0 & \frac{P}{2}\left(Z - \frac{1}{K^2 Z}\right) \\ -\frac{P}{L-2Z} & 0 & \frac{P}{L-2Z} & 0 \\ 0 & \frac{P}{2}\left(Z - \frac{1}{K^2 Z}\right) & 0 & \frac{P}{2}\left(Z + \frac{1}{K^2 Z}\right) \end{bmatrix} \begin{Bmatrix} v_3 \\ \theta_3 \\ v_4 \\ \theta_4 \end{Bmatrix}. \tag{27}$$

In this expression, the translational and rotational stiffness terms are completely decoupled.

3.6. Deviation from a Pure Pendulum

The flexure rod is not identical in all respects to a pin-ended virtual pendulum of length $L - 2Z$. The differences can be seen in the stiffness matrix of equation (27).

If the rod behaved like a pure pendulum, the rotational stiffness terms $(P/2)(Z \pm 1/K^2 Z)$ would not be present (i.e., there would be no resistance to local rotation about the virtual pivot points). For the actual rod, the rotational stiffness terms have an effect equivalent to a rotational spring with stiffness $(P/2)(-Z + 1/K^2 Z)$ connecting the two pivot points, in combination with rotational springs with stiffness PZ connecting each pivot point to ground.

3.7. Transformation to Lower Zero Moment Point

The design of the two-stage isolation system envisions all lateral actuation forces occurring at the lower zero moment point of the flexure rods. Thus, the coordinates in Figure 3c are most appropriate. The corresponding stiffness matrix is obtained by making the following change of variables

$$\begin{aligned} v_4 &= v_5 - (L - 2Z)\theta_5 & V_4 &= V_5 \\ \theta_4 &= \theta_5 & M_4 &= (L - 2Z)V_5 + M_5 \end{aligned} \tag{28}$$

in equation (27). The 5 subscript refers to displacements, rotations, forces, and moments at the lower zero moment point, rigidly transferred to the upper end. The resulting stiffness matrix is

$$\begin{Bmatrix} V_3 \\ M_3 \\ V_5 \\ M_5 \end{Bmatrix} = \begin{bmatrix} \frac{P}{L-2Z} & 0 & -\frac{P}{L-2Z} & P \\ 0 & \frac{P}{2}\left(Z + \frac{1}{K^2 Z}\right) & 0 & \frac{P}{2}\left(Z - \frac{1}{K^2 Z}\right) \\ -\frac{P}{L-2Z} & 0 & \frac{P}{L-2Z} & -P \\ P & \frac{P}{2}\left(Z - \frac{1}{K^2 Z}\right) & -P & \frac{P}{2}\left(Z + \frac{1}{K^2 Z}\right) + P(L - 2Z) \end{bmatrix} \begin{Bmatrix} v_3 \\ \theta_3 \\ v_5 \\ \theta_5 \end{Bmatrix}. \tag{29}$$

The first column of this matrix implies that a pure translation at the lower zero moment requires a force $Pv_3/(L-2Z)$, and this force is reacted to the upper support along with a moment Pv_3. (This can also be thought of as the moment resulting from the translated gravity load.) *The reacted moment will result in tilt of the system during lateral actuation, as the moment deforms the vertical springs of the supporting suspension.* This is true even for the ideal case of actuating directly through the lower zero moment point.

4. Stress Analysis

Under the assumptions of slender beam bending theory, the stress state in the rod at any station is a superposition of shear stress (which varies quadratically across the cross section) with axial stress (which varies linearly across the cross section). The total total bending moment M and shear Q in the rod at station x are obtained by differentiating equation (6):

$$M(x) = EI\frac{d^2v}{dx^2} = \frac{EI}{2}\left[\left(v_1 - v_2 + (\theta_1+\theta_2)\frac{L}{2}\right)\frac{K^2\sinh K\left(x-\frac{L}{2}\right)}{\frac{KL}{2}\cosh\frac{KL}{2} - \sinh\frac{KL}{2}} + (\theta_2-\theta_1)\frac{K\cosh K\left(x-\frac{L}{2}\right)}{\sinh\frac{KL}{2}}\right], \tag{30}$$

$$Q(x) = -\frac{d}{dx}\left(EI\frac{d^2v}{dx^2}\right) = -\frac{EI}{2}\left[\left(v_1 - v_2 + (\theta_1+\theta_2)\frac{L}{2}\right)\frac{K^3\cosh K\left(x-\frac{L}{2}\right)}{\frac{KL}{2}\cosh\frac{KL}{2} - \sinh\frac{KL}{2}} + (\theta_2-\theta_1)\frac{K^2\sinh K\left(x-\frac{L}{2}\right)}{\sinh\frac{KL}{2}}\right]. \tag{31}$$

To first order, the axial load in the rod is P at all stations.

From equations (30) and (31), it is clear that both shear and moment in the rod are maximized at the ends ($x=0$ and $x=L$). After simplification (such as replacing EIK^2 with P), the moment and shear at the two ends are:

$$M(0) = -\frac{PZ}{L-2Z}\left(v_1 - v_2 + (\theta_1+\theta_2)\frac{L}{2}\right) + \frac{EI}{2Z}(\theta_2-\theta_1)\,, \tag{32}$$

$$M(L) = \frac{PZ}{L-2Z}\left(v_1 - v_2 + (\theta_1+\theta_2)\frac{L}{2}\right) + \frac{EI}{2Z}(\theta_2-\theta_1)\,, \tag{33}$$

$$Q(0) = \frac{P}{L-2Z}\left(v_1 - v_2 + (\theta_1+\theta_2)\frac{L}{2}\right) - \frac{P}{2}(\theta_2-\theta_1)\,, \tag{34}$$

$$Q(L) = \frac{P}{L-2Z}\left(v_1 - v_2 + (\theta_1+\theta_2)\frac{L}{2}\right) + \frac{P}{2}(\theta_2-\theta_1)\,. \tag{35}$$

The rod will be sized to achieve the desired lateral stiffness, while keeping the stress within requirements. The design condition for stress evaluation is a combination of the dead load P, together with bending stress associated with a lateral displacement δ (currently δ is 1 mm). This scenario is obtained by setting $v_1 = \delta$, $v_2 = 0$, and $\theta_1 = \theta_2 = 0$ in the above equations.

The average shear stress at the end is given by $\tau = Q/A$, where A is the cross sectional area of the rod:

Average Shear Stress:
$$\boxed{\tau_{\max} = \frac{P\delta}{A(L-2Z)}} \tag{36}$$

The peak shear stress will occur at the neutral axis. Also, since $\delta \ll (L - 2Z)$, the shear stress is much smaller than the P/A axial stress, and can therefore be ignored for initial design purposes.

The axial stress at a distance y from the neutral axis is given by

$$\sigma = \frac{P}{A} + \frac{My}{I} \; . \tag{37}$$

Substituting the end moment from equation (32) and choosing y as the maximum fiber distance c, we obtain

Max Axial Stress:
$$\boxed{\sigma_{\max} = \frac{P}{A} + \frac{PZc\delta}{I(L-2Z)}} \tag{38}$$

For the special case of a circular cross section of diameter D, we have $A = (\pi/4)D^2$, $I = (\pi/64)D^4$, and $c = D/2$. Then equation (38) becomes

$$\boxed{\sigma_{\max} = \frac{4}{\pi}\,\frac{P}{D^2}\left(1 + \frac{8Z\delta}{D(L-2Z)}\right)} \tag{39}$$