Formal studies of different interferometer noises

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- Studies of non Gaussian tails and their effects on the non linearities of seismic isolation systems. The future aim is to estimate the relevance of noise bursts in the seismic noise measurements.

- Useful techniques to study non-linear regime systems borrowed from field theory physics.

- Random walk simulations of Brownian motion models whose low frequency spectral behaviour show tails similar to those observed in mirror internal thermal noise spectra.

Statistics in the time domain to characterize the peaks of noise affecting the Seismic Attenuation System

Motivation

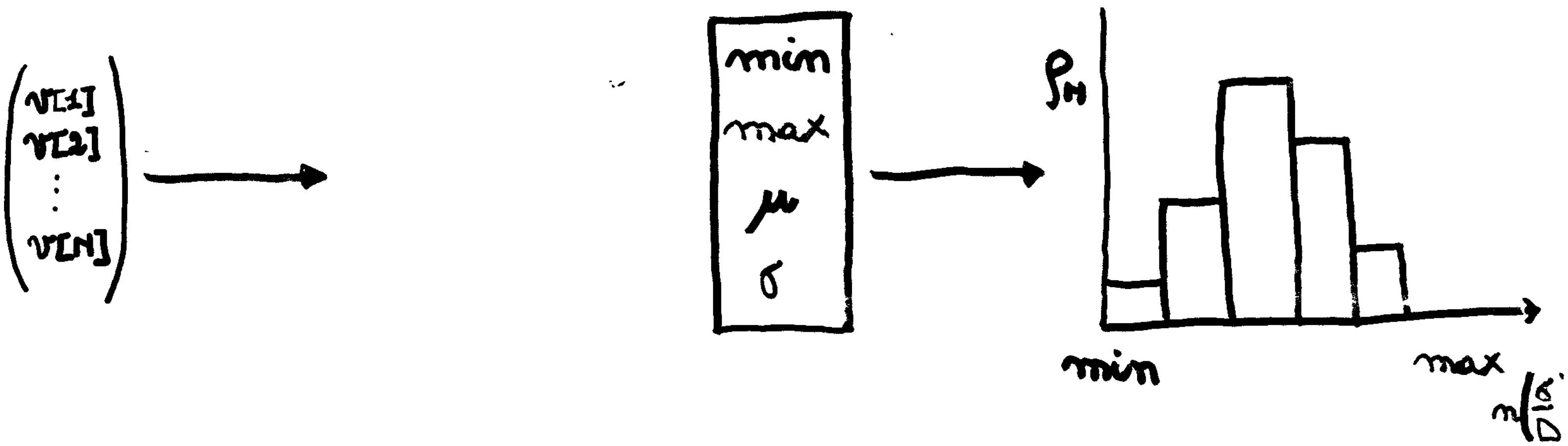
The variations of the length dividing two mirros is measured through a laser interferometric technique. Every motion of the masses is a limit to the observation of astrophysical signals. In order to cut down the noise active and passive systems are designed.

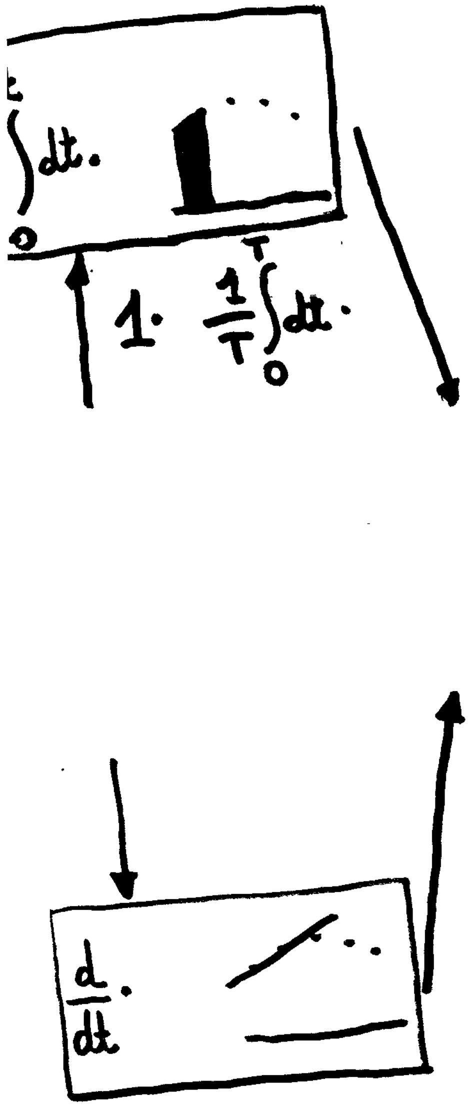
Methods

- Acquisition system (accelerometers and spectral analyser).
- C code using mathematical and Numerical Recipes libraries.

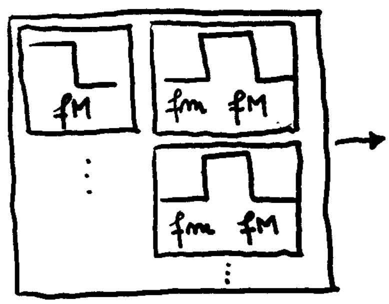
Aims

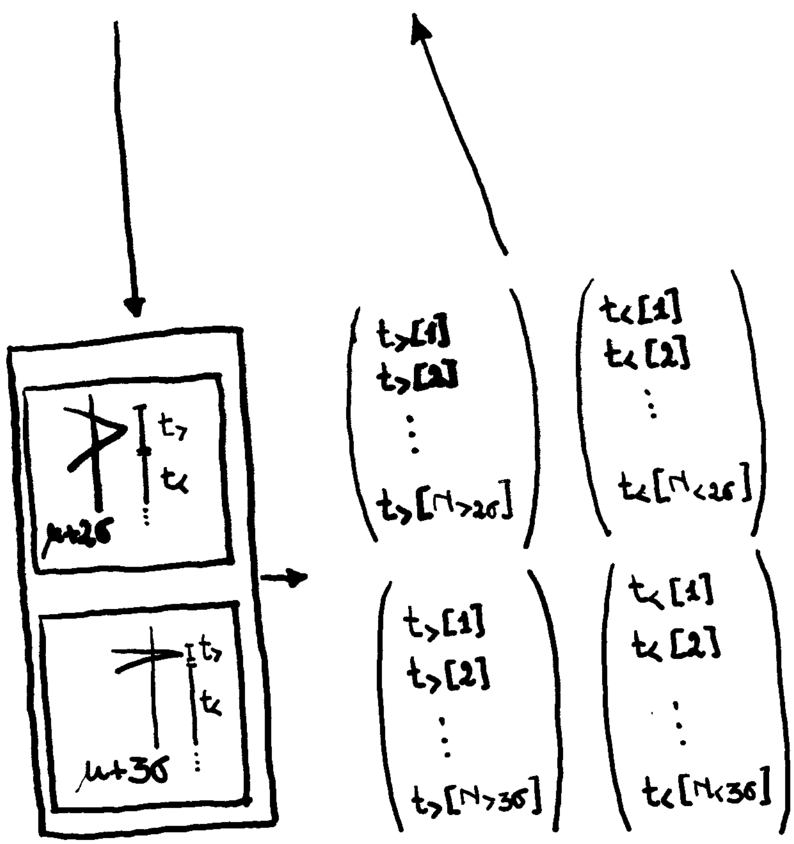
- study of the distribution of noise measurements around the average value.
- application of frequency filters.
- characterization of the time intervals between peaks and their durations.

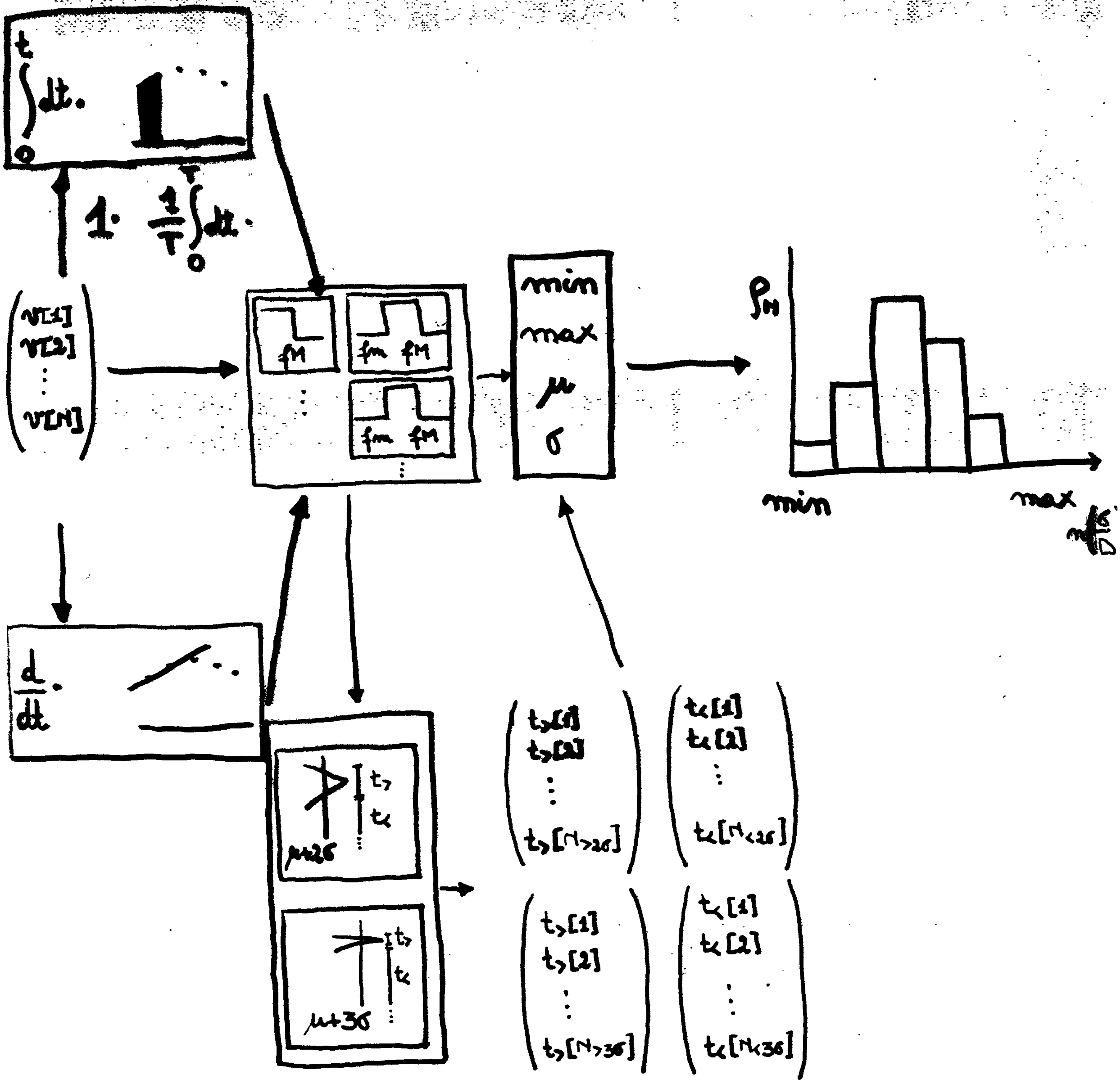

v[1]
v[2]
v[N]
min
max
μ
σ
min
max



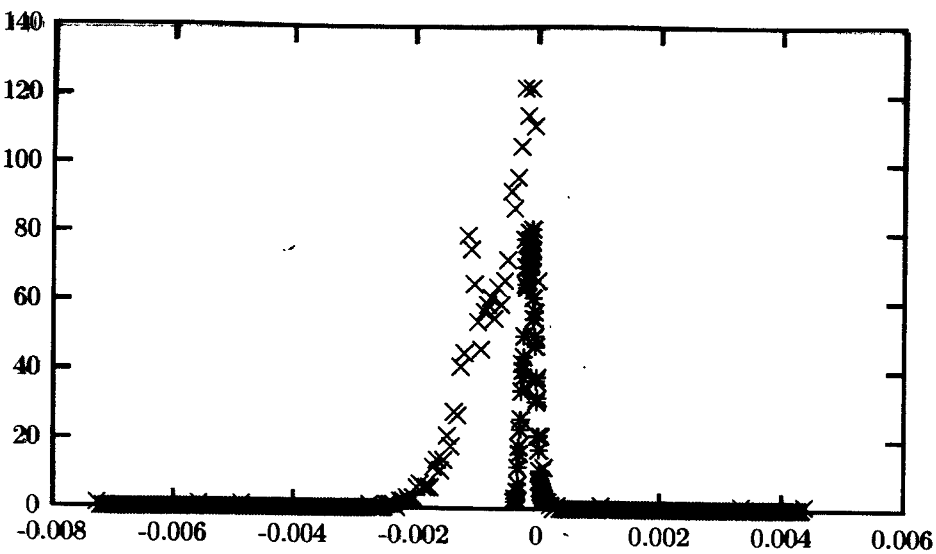
dt.
1. 1/T ∫ dt.
d/dt.


fM
fm fM
fm fM


μ+2σ
t>
t<
μ+3σ
t>
t<
t>[1]
t>[2]
t>[N>2σ]
t<[1]
t<[2]
t<[N<2σ]
t>[1]
t>[2]
t>[N>3σ]
t<[1]
t<[2]
t<[N<3σ]



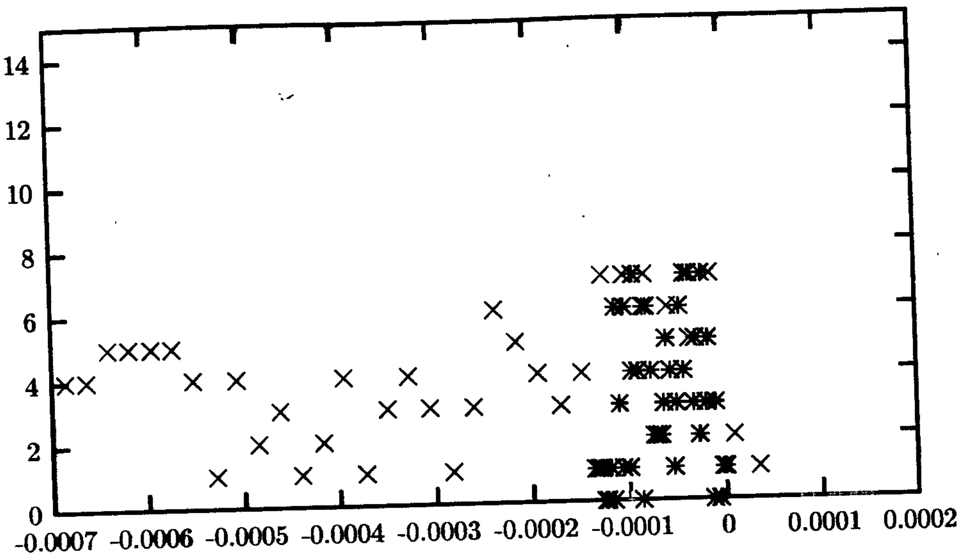
v[1]
v[2]
v[N]
fM
fm fM
fm fM
min
max
μ
σ
ρM
min
max
μ+2σ
μ+3σ
t>
t<
t>[1]
t>[2]
t<[1]
t<[2]

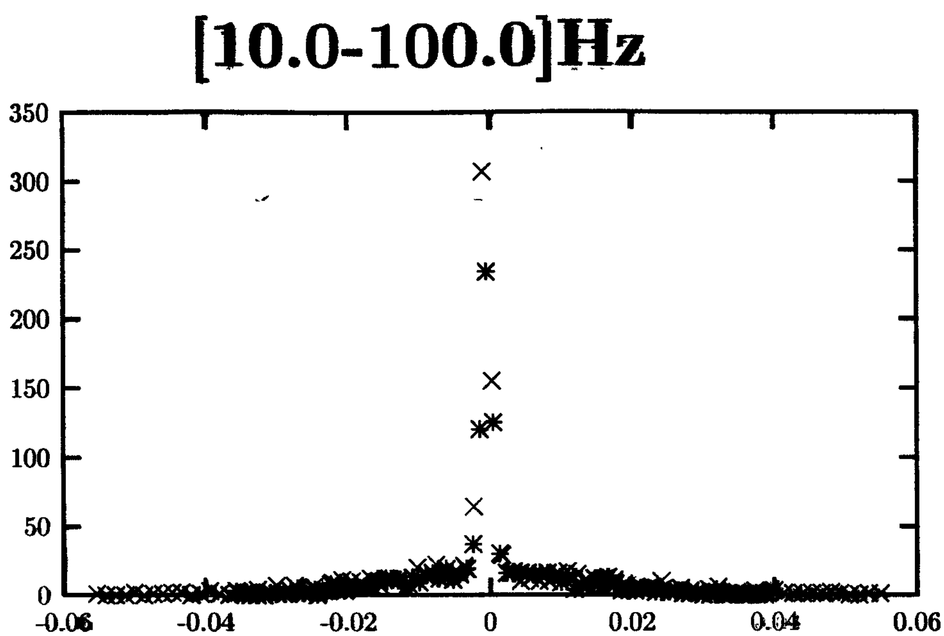


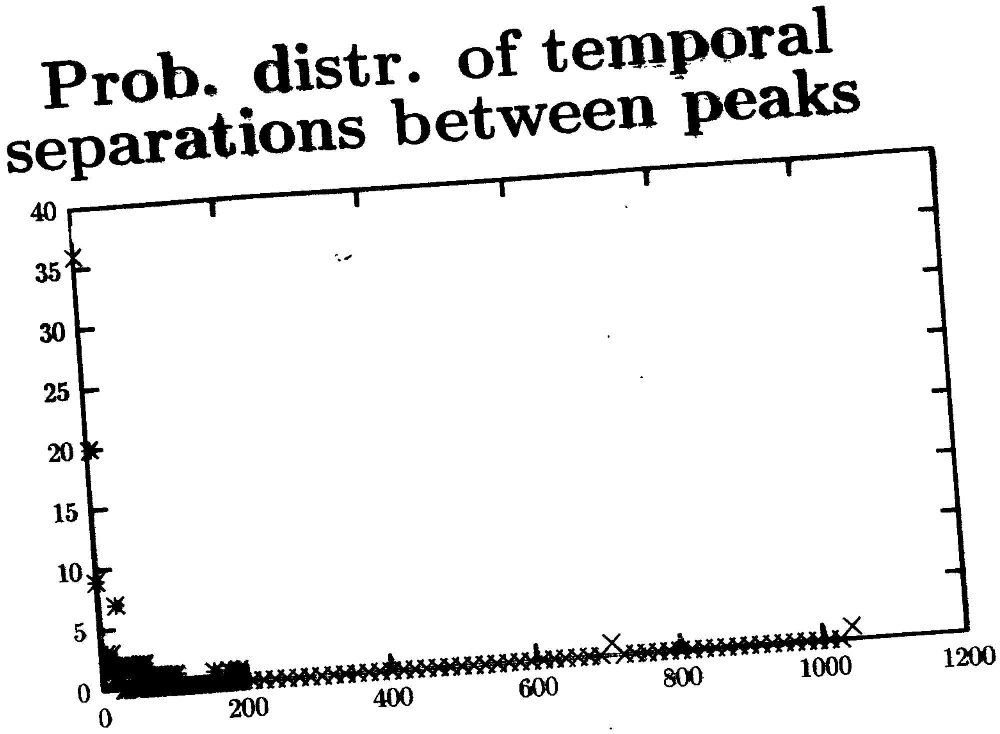
2048 *points*

256 Hz

[0-10.0]Hz

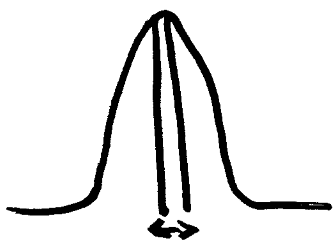

14
12
10
8
6
4
2
0
-0.0007
-0.0006
-0.0005
-0.0004
-0.0003
-0.0002
-0.0001
0
0.0001
0.0002


[10.0-100.0]Hz
350
300
250
200
150
100
50
0
-0.06
-0.04
-0.02
0
0.02
0.04
0.06

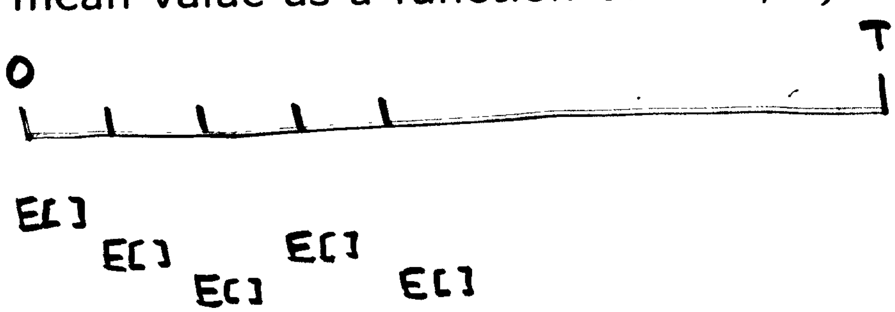

Prob. distr. of temporal separations between peaks
40
35
30
25
20
15
10
5
0
0
200
400
600
800
1000
1200

Work in progress ...

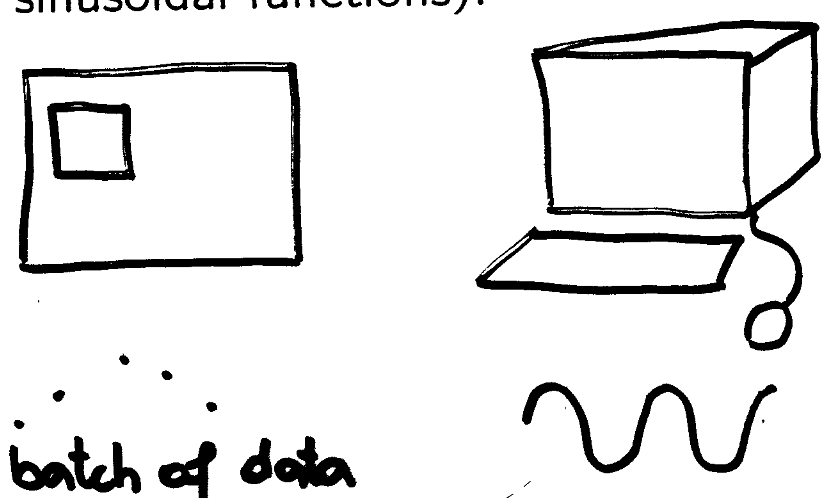
- Estimation of higher order moments (skewness, ...) to verify gaussianity.



- Subdivision of the data array in smaller chunks with a view to tell some trends (drift of the mean value as a function of time,...).



- Point out possible periodicities (convolution with sinusoidal functions).



Non-linear Effects in Stationary Stochastic Systems Dynamics

Motivation

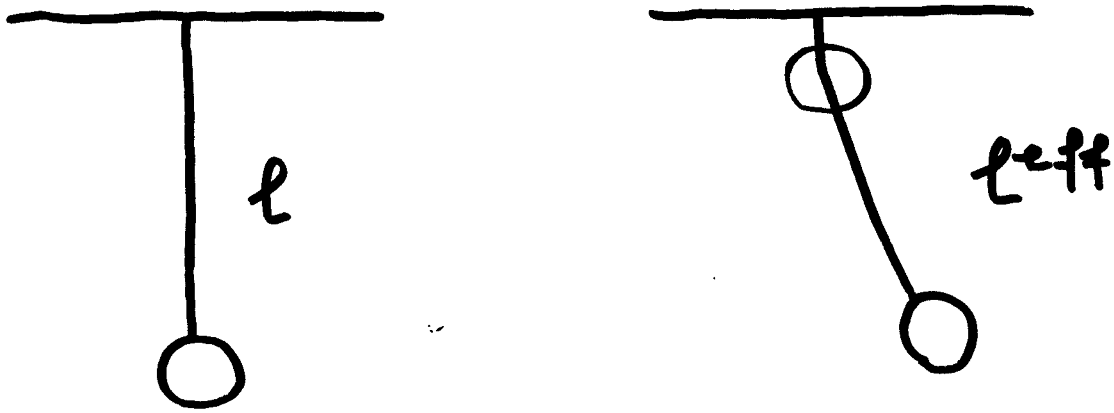
Investigation of the relevance of small non-linear effects and their impact on the sensitivity curve of interferometric gravitational wave detectors.

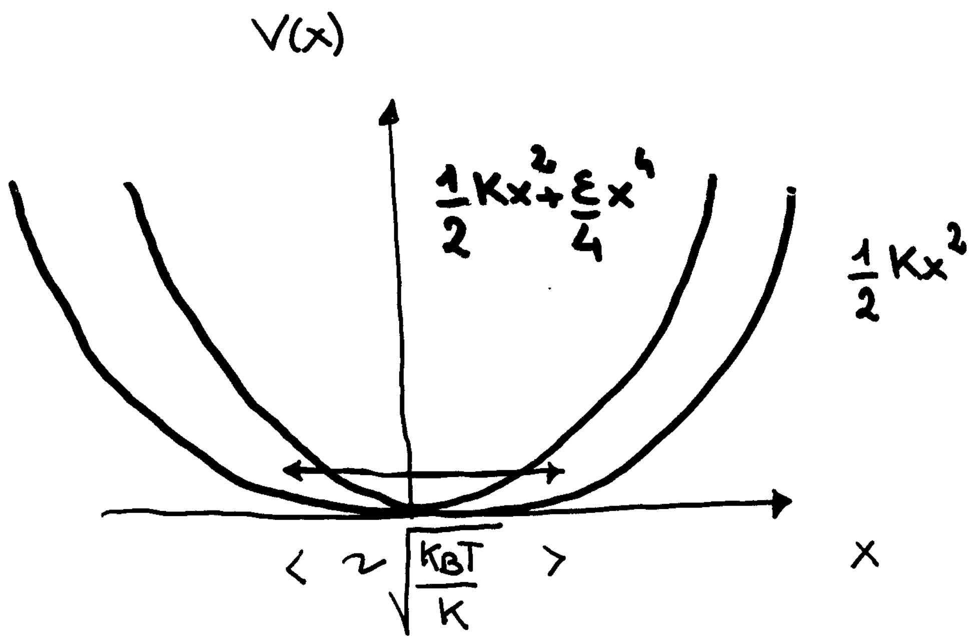
Methods

- Perturbative techniques using Feynman diagrams and renormalization rules
- Numerical simulations ("Simulation of Supersymmetric Models with a ..." Phys. Rev. D Vol. 58 (1998) ID 065009).

Results

- Application of the sum rules borrowed from field theory physics in order to define renormalized coefficients.
- Negligible variations if the fluctuations are only due the thermal noise.
- Legitimate issues about the seismic noise spectra al least for the VIRGO SuperAttenuator


ℓ
ℓeff


V(x)
$\frac{1}{2}Kx^2 + \frac{\varepsilon}{4}x^4$
$\frac{1}{2}Kx^2$
$< \sim \sqrt{\frac{k_B T}{K}} >$
x

$$K^{eff} \sim V''(x)$$

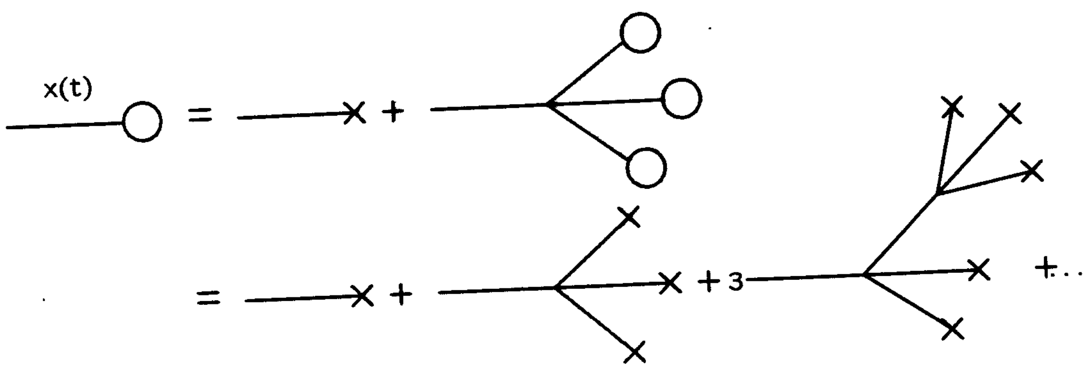
Graphical representation of perturbative calculus

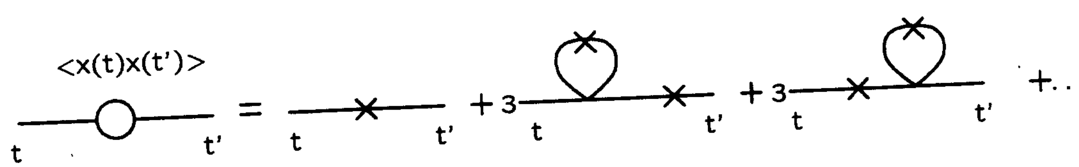
$$m\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) = \xi(t)$$

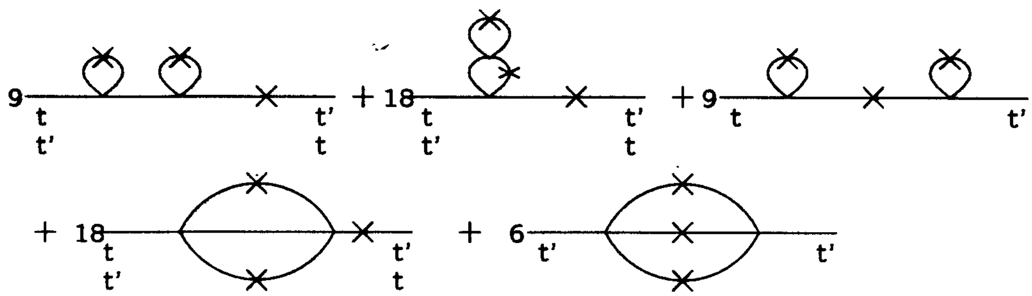
$$< \xi(t)\xi(t') >= 2\gamma K_B T \delta(t - t')$$

$$m\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) + \epsilon x^3 = \xi(t)$$

$$\mathcal{G}(t) = \frac{\theta(t)}{m} e^{-\frac{\gamma}{2m}t} \frac{\sin\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}t}{\sqrt{\frac{k}{m} - \frac{\gamma^2}{4m^2}}}$$







$$S(f) = \frac{\frac{2\gamma K_B T}{m^2}}{[\frac{k}{m}(1+\frac{3\alpha}{2}-9\alpha^2)-\omega^2]^2+\frac{\omega^2\gamma^2}{m^2}(1-\frac{27}{2}\frac{mk}{\gamma^2}\alpha^2)}$$

$$+ \frac{\frac{2\gamma K_B T}{m^2}}{([\frac{k}{m}-\omega^2]^2+\frac{\omega^2\gamma^2}{m^2})^2}\frac{\frac{(3k\gamma\alpha)^2}{2m^4}}{\frac{k}{m}-\frac{\gamma^2}{4m^2}} \times$$

$$\left[\frac{\omega^2(\frac{\gamma^2}{m^2}-12\frac{k}{m})+2\frac{\gamma^2}{m^2}(\frac{k}{m}+2\frac{\gamma^2}{m^2})}{(\omega^2-(\frac{k}{\gamma}+2\frac{\gamma^2}{m^2}))^2+(\frac{3\omega\gamma}{m})^2} + \frac{4\frac{k^2}{\gamma^2}(8\omega-36\frac{k}{m}-9\frac{\gamma^2}{m^2})}{(\omega^2-\frac{9k}{m})^2+(\frac{3\gamma\omega}{m})^2}\right]$$

$$kx \sim \epsilon x^3 \Rightarrow k \sim \epsilon < x^2 >$$

$$< x^2 > = \frac{K_B T}{k} \qquad \frac{\alpha}{2} = \frac{K_B T}{k^2}\epsilon$$

Comparison with the Boltzmann distribution

$$S(f) = \int_{-\infty}^{+\infty} e^{-2\pi i f \tau} < x(t)x(t+\tau) > \mathrm{d}\tau$$

and from this definition

$$< x^2(t) >= (\frac{K_B T}{k})(1 - \frac{3\alpha}{2} + 6\alpha^2)$$

which is equivalent to

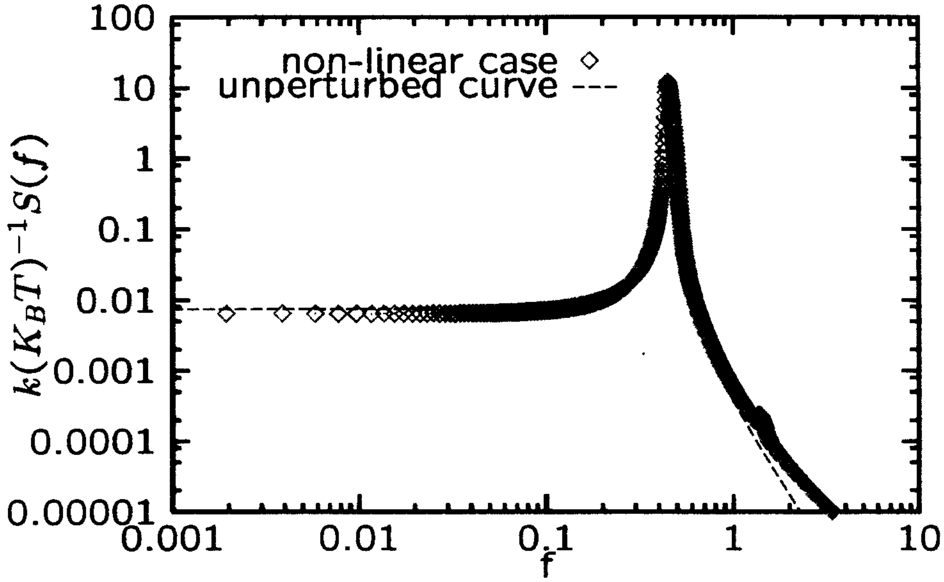
$$\begin{aligned} < x^2 > &= (\frac{K_B T}{k}) \frac{\int_{-\infty}^{+\infty} e^{-(\frac{x'^2}{2}+\frac{\alpha}{8}x'^4)} x'^2 \mathrm{d}x'}{\int_{-\infty}^{+\infty} e^{-(\frac{x'^2}{2}+\frac{\alpha}{8}x'^4)} \mathrm{d}x'} \\ &= (\frac{K_B T}{k})(1 - \frac{3\alpha}{2} + 6\alpha^2) . \end{aligned}$$

Moments of order greater than two are no longer equal to zero and we find that

$$< x^4(t) >_{conn} = -3\alpha(\frac{K_B T}{k})^2 + 14\alpha^2(\frac{3K_B T}{2k})^2$$

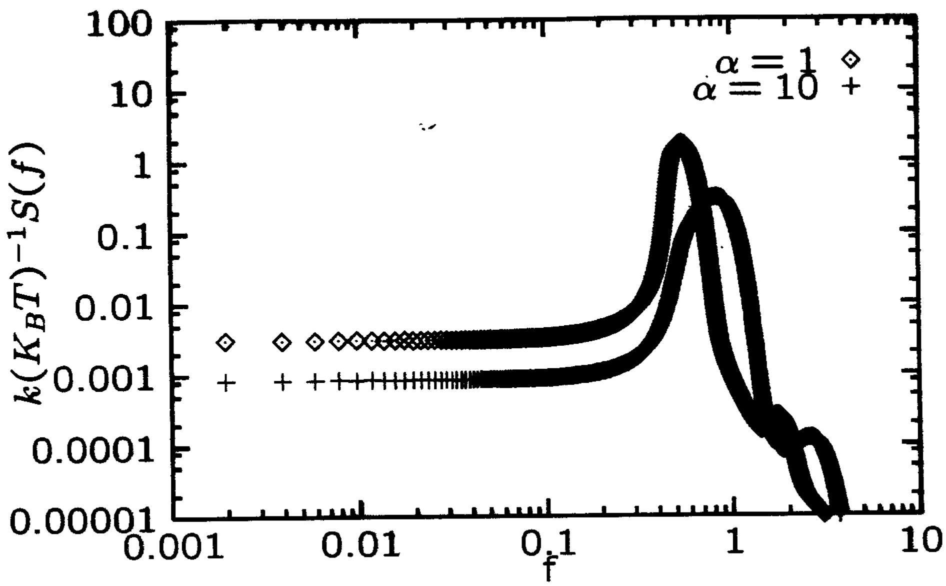
to be compared to

$$< x^4 > -3 < x^2 >< x^2 > = (\frac{K_B T}{k})^2(-3\alpha + (\frac{63}{2})\alpha^2)$$



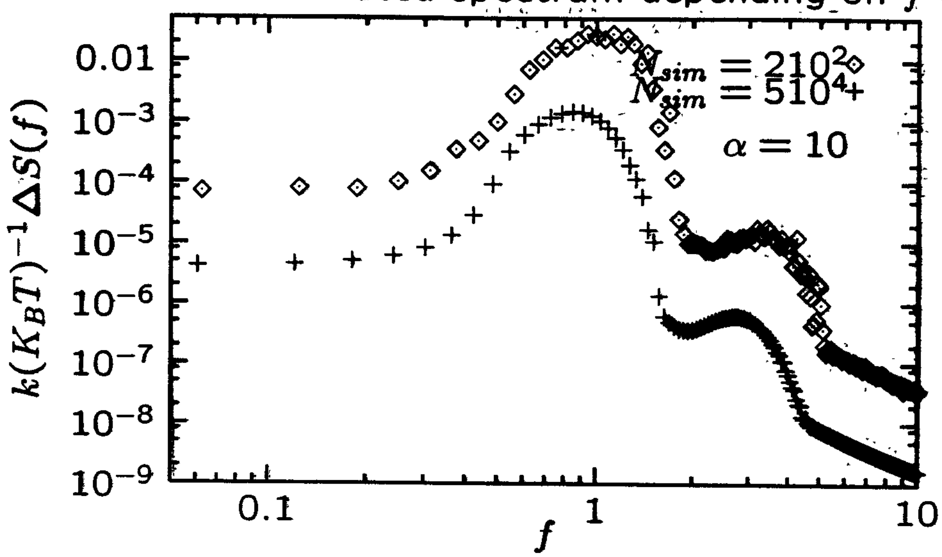
non-linear case
unperturbed curve
100
10
1
0.1
0.01
0.001
0.0001
0.00001
0.001
0.01
0.1
1
10
f
$k(K_BT)^{-1}S(f)$

7

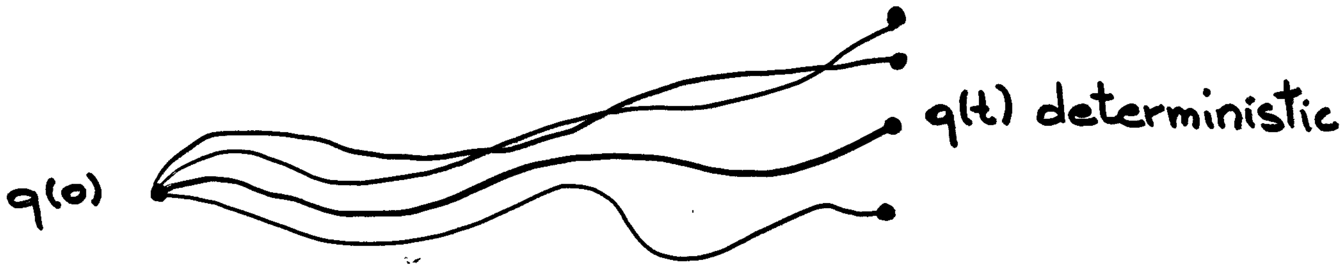


100
10
1
0.1
0.01
0.001
0.0001
0.00001
0.001
0.01
0.1
1
10
f
$k(K_BT)^{-1}S(f)$
$\alpha = 1$
$\alpha = 10$

Errors on the estimated spectrum depending on f and N_{sim}



$$D \longleftrightarrow \hbar$$



Stochastic Quantization

$$\dot{q} = f(q) + \sqrt{2D}\xi$$

$$< \xi(t) >= 0 \qquad\qquad < \xi(t)\xi(t') >= \delta(t - t')$$

$$H = -D\frac{\partial^2}{\partial q^2} + \frac{\partial}{\partial q} f(q) \quad .$$

$$\begin{aligned} Z(J,K) &= \int < q, t_f | \mathcal{U}_{JK}(t_f, t_i) | q_i, t_i > \mathrm{d}q \\ \mathcal{U}_{JK}(t_f, t_i) &= \mathrm{T}\exp \int_{t_f}^{t_i} (J(t)\widehat{q}(t) - iK(t)\widehat{p}(t))\mathrm{d}t \end{aligned}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta Z(J,K)}{\delta J(t)} = [K(t) + 2D\frac{\delta}{\delta K(t)} + f(\frac{\delta}{\delta J(t)})]Z(J,K)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta Z(J,K)}{\delta K(t)} = \\ [\frac{1}{2}f''(\frac{\delta}{\delta J(t)}) - J(t) - f'(\frac{\delta}{\delta J(t)})\frac{\delta}{\delta K(t)}]Z(J,K)$$

$$Z(J,K) = \\ < \exp \int_{t_i}^{t_f} (J(t)q(t) - \frac{K^2(t)}{4D} + K(t)\frac{\dot{q} - f(q)}{\sqrt{2D}})\mathrm{d}t >$$

Sum Rules and Renormalization Approach

The elastic constant becomes

$$m\omega_0^2 \to m\omega_0^2 + \frac{3K_BT\epsilon}{m\omega_0^2 + \frac{3K_BT\epsilon}{m\omega_0^2+\dots}} = m\omega_0^2(1+\frac{3\alpha}{2}-(\frac{3\alpha}{2})^2+\dots) \quad (1)$$

like a recursive definition.

It corresponds to the recursive expansion in tadpole type bubbles of a tadpole.

Let us call renormalized the tadpole with all tadpole corrections included and then use this definition to write the corrections to the bare tadpole. We treat this implicit definition using the field theory tools applied to the geometric sums of Feynman diagrams.

With the same language of field theory we may define a renormalized Ω_0.

The final result is

$$\Omega_0^2 = \frac{\omega_0^2}{2}[1 + \sqrt{1 + \frac{12K_BT\epsilon}{(m\omega_0^2)^2}}] = \frac{\omega_0^2}{2}[1 + \sqrt{1 + 6\alpha}] \quad .$$

In order to understand such a modification we may note that there are stronger forces pulling the system back to the equilibrium configuration and the result is that the measured recall constant $m\omega_0^2$ has a new greater value.

("Perturbative and Numerical Methods for ..." accepted by Physica A-Elsevier)

This is an important physical result: we may use the transfer functions written in the linear hypothesis.

We recall that the active control of gravitional wave detectors are completely based on the transfer functions. We have only to insert the real parameters and not the bare ones.

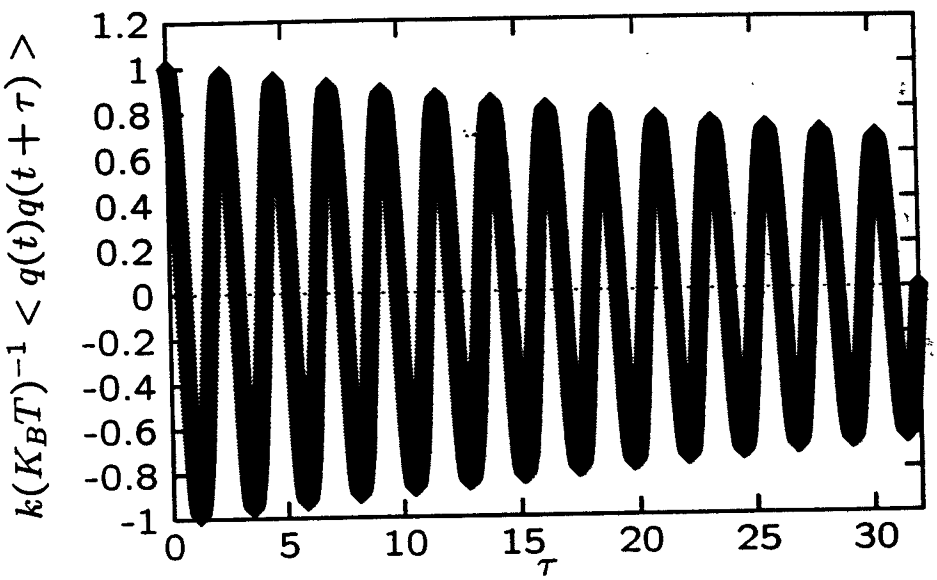
We expect that the resonant frequency should be greater if α goes up. It means that the system explores the non-linear zone of the phase space. When T rises the greater amplitude of fluctuations just produce such an effect. If the recall constant increases it is obvious that the opposite tendency is obtained.

Numerical Simulation

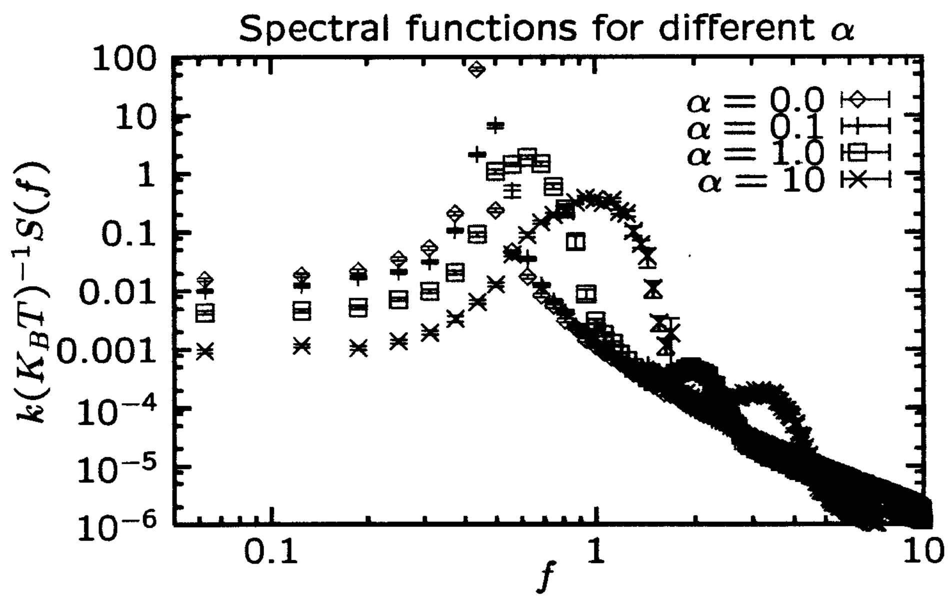
$$P(q'',p'',t''|q',p',t') =< q'',p''|U(t'',t')|q',p' >$$

$$P(q'',p'',t''|q',p',t') = \int \mathrm{d}q_1 \int \mathrm{d}q_2 \int \mathrm{d}p_1 \int \mathrm{d}p_2$$
$$< q'',p''|e^{-\frac{t''-t'}{2}\hat{H}_1}|q_2,p_2 >< q_2,p_2|e^{-(t''-t')\hat{H}_2}|q_1,p_2 > \times$$
$$< q_1,p_2|e^{-\frac{t''-t'}{2}\hat{H}_1}|q',p' > + O\left((t''-t')^3\right)$$

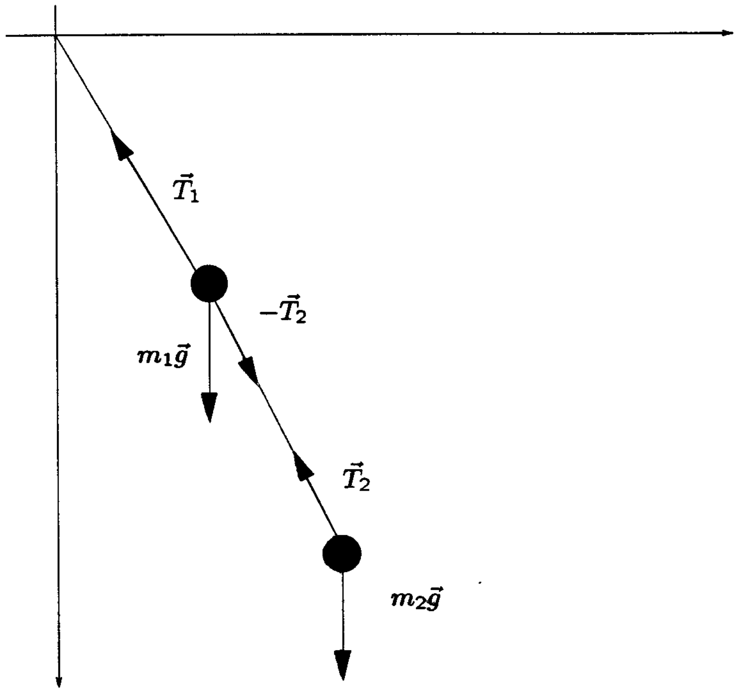
$$\begin{cases} p_1 = p' + \frac{\Delta t}{2}f(q') + \sqrt{D\Delta t}\xi_1 \\ q_1 = q' \end{cases}$$
$$\Downarrow$$
$$\begin{cases} p_2 = (1 - \frac{\gamma\Delta t}{m} + \frac{1}{2}(\frac{\gamma\Delta t}{m})^2)p_1 \\ q_2 = q_1 + \frac{p_1}{\gamma}(\frac{\gamma\Delta t}{m} - \frac{1}{2}(\frac{\gamma\Delta t}{m})^2) \end{cases}$$
$$\Downarrow$$
$$\begin{cases} p'' = p_2 + \frac{\Delta t}{2}f(q_2) + \sqrt{D\Delta t}\xi_2 \\ q'' = q_2 \end{cases}$$



$k(K_BT)^{-1} < q(t)q(t+\tau) >$
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
0
5
10
15
20
25
30
τ



Spectral functions for different α
$\alpha = 0.0$
$\alpha = 0.1$
$\alpha = 1.0$
$\alpha = 10$
100
10
1
0.1
0.01
0.001
10^{-4}
10^{-5}
10^{-6}
0.1
1
10
f
$k(K_BT)^{-1}S(f)$

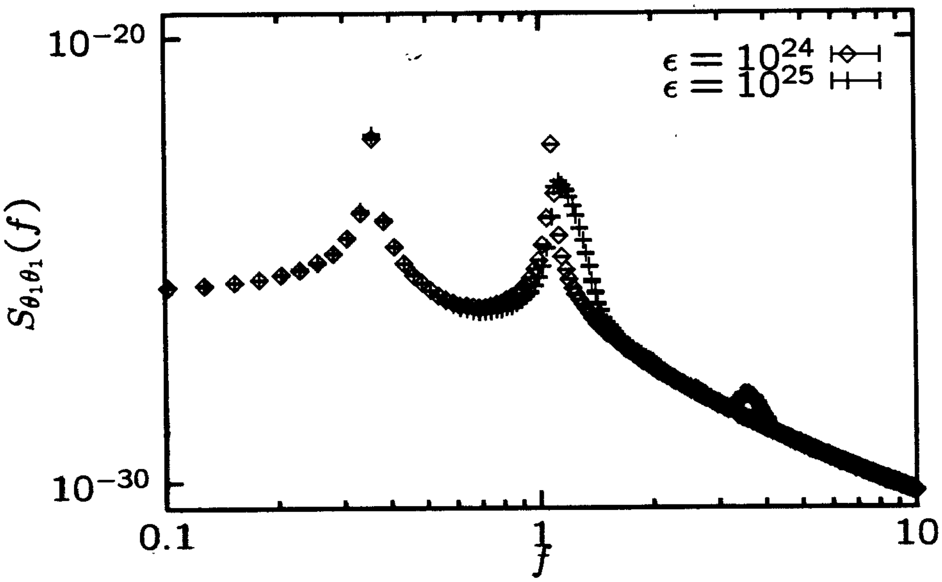


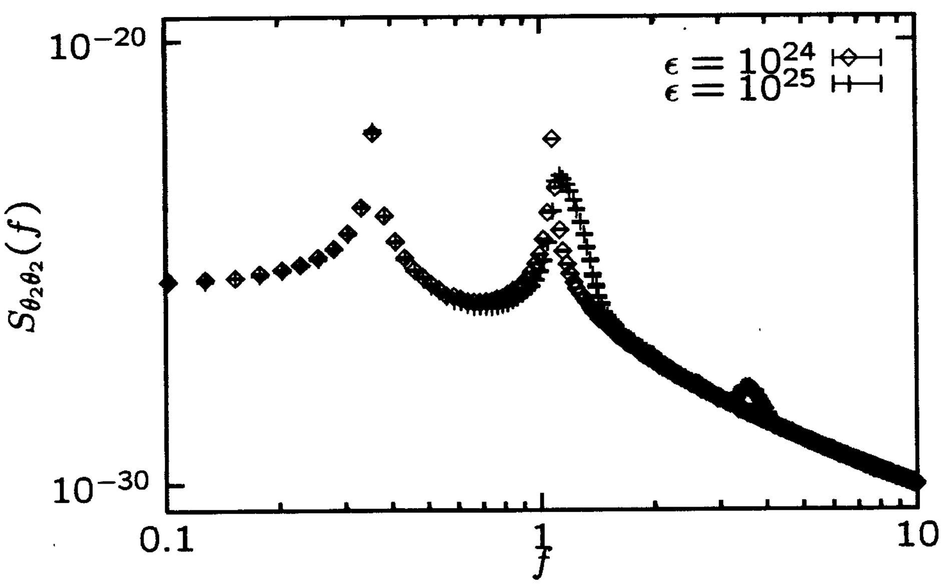
$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{R}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{R}}_2^{\,2} + m_1 \vec{g} \cdot \vec{R}_1 + m_2 \vec{g} \cdot \vec{R}_2$$

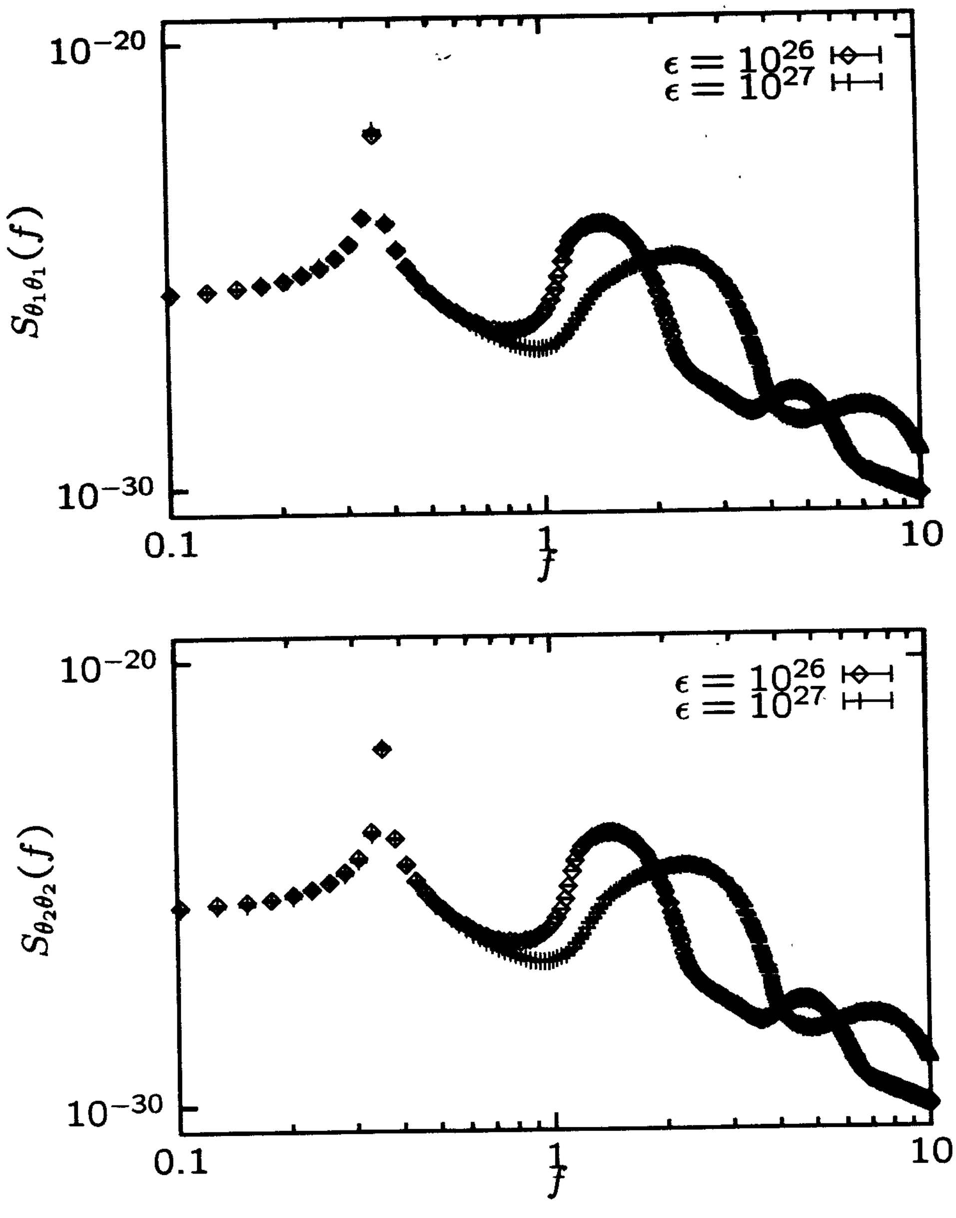
$$m_2 \ddot{\vec{R}}_2 = m_2 \vec{g} + \vec{T}_2 \qquad m_1 \ddot{\vec{R}}_1 = m_1 \vec{g} - \vec{T}_2 + \vec{T}_1$$

$$\begin{aligned} m_2 \ddot{\vec{R}}_2 &= m_2 \vec{g} + \vec{T}_2 - \gamma_2 \dot{\vec{R}}_2 + \vec{F}_2^S \\ m_1 \ddot{\vec{R}}_1 &= m_1 \vec{g} - \vec{T}_2 + \vec{T}_1 - \gamma_1 \dot{\vec{R}}_1 + \vec{F}_1^S \end{aligned}$$

$$V_I = \frac{\epsilon}{4}(\theta_1 - \theta_2)^4$$

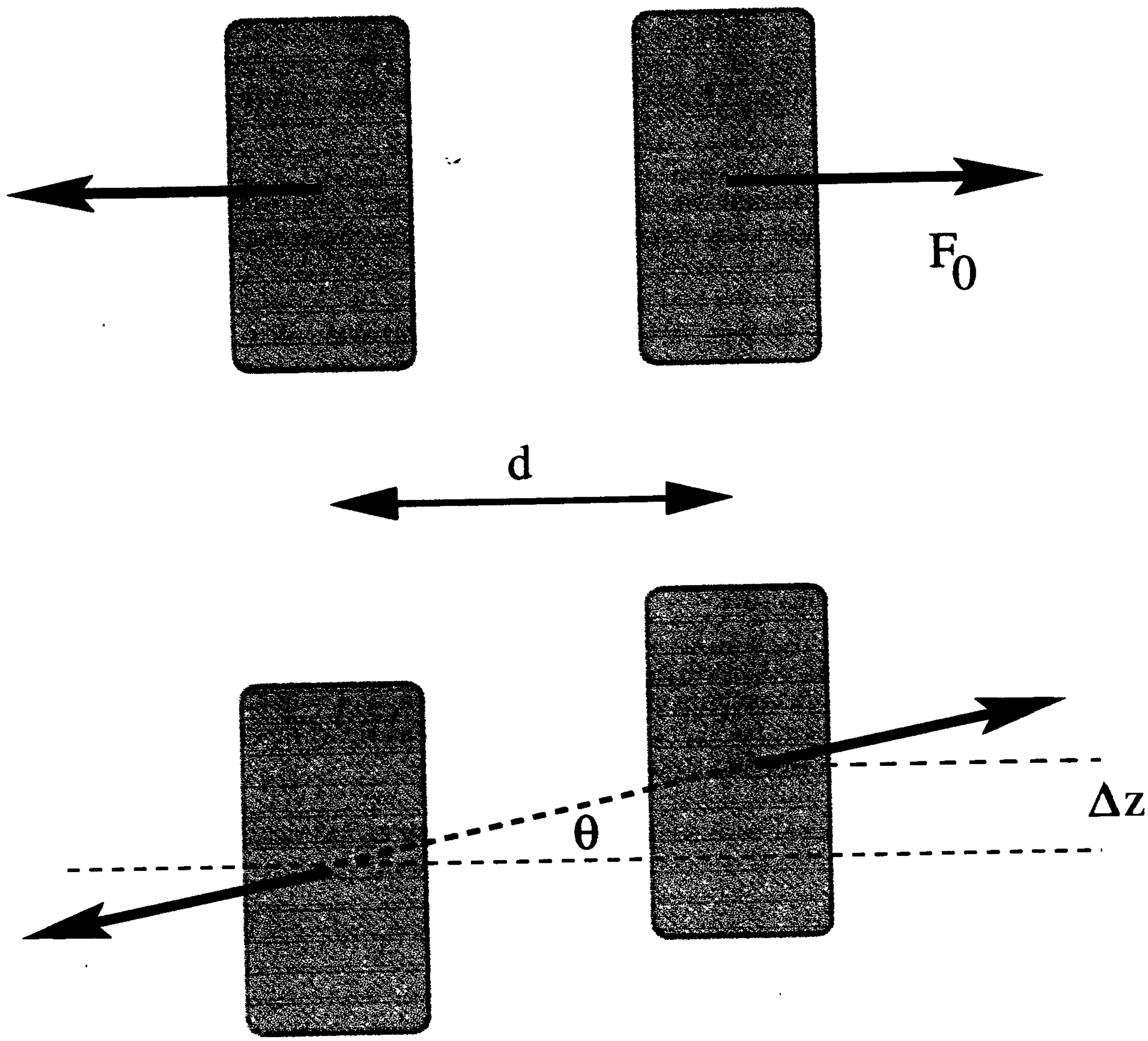






10^{-20}
10^{-30}
$\epsilon = 10^{26}$
$\epsilon = 10^{27}$
$S_{\theta_1\theta_1}(f)$
0.1
1
10
f
10^{-20}
10^{-30}
$\epsilon = 10^{26}$
$\epsilon = 10^{27}$
$S_{\theta_2\theta_2}(f)$
0.1
1
10
f

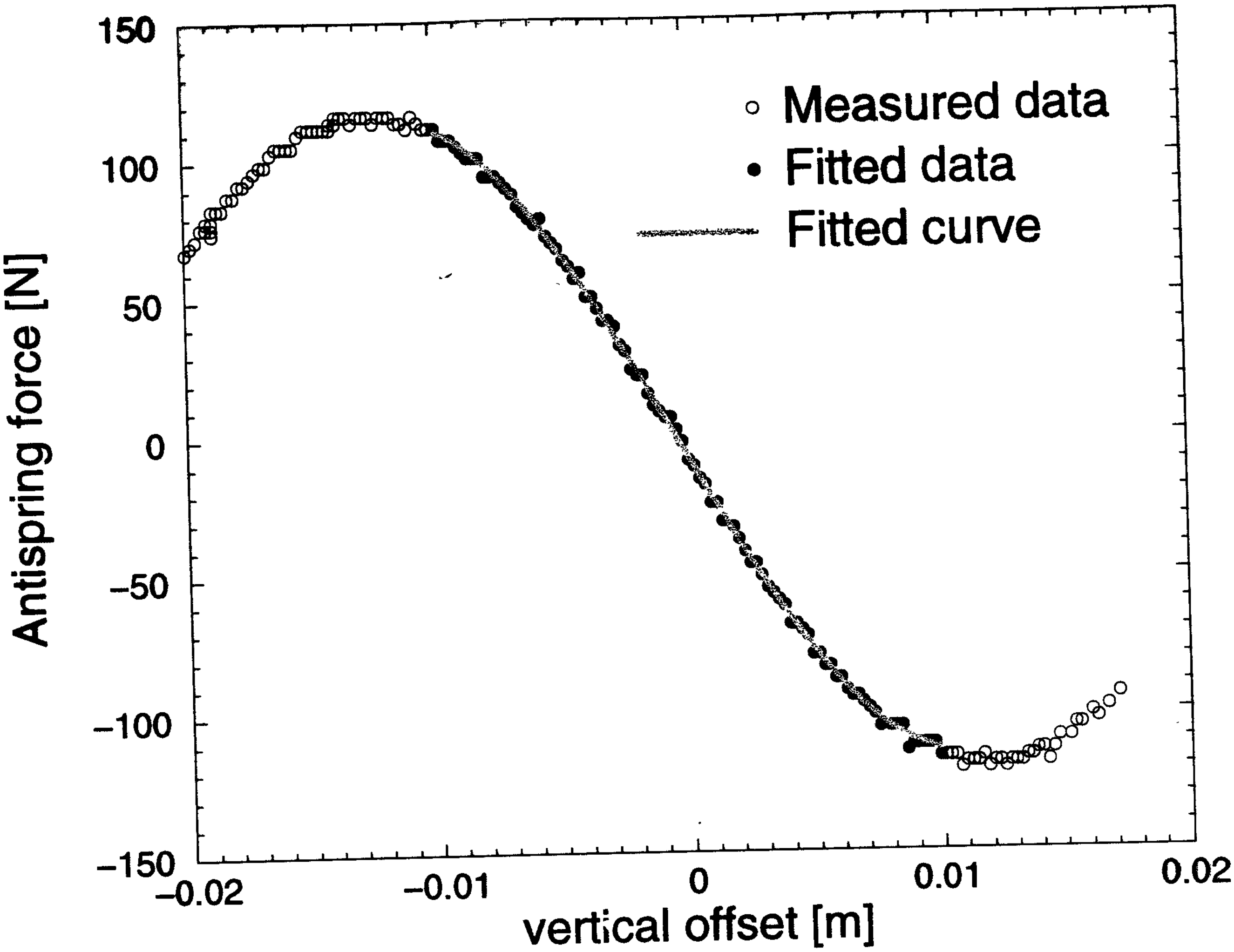
Non-linearities in antisprings



$$F_z = F_0 \sin\theta \simeq F_0 \frac{\Delta z}{d}[1 + \ldots]$$

$$k_{tot} = k_{blades} - \frac{F_0}{d}$$

$$k_{blades} \simeq \frac{F_0}{d} \simeq 1.5\, 10^4 N/m$$

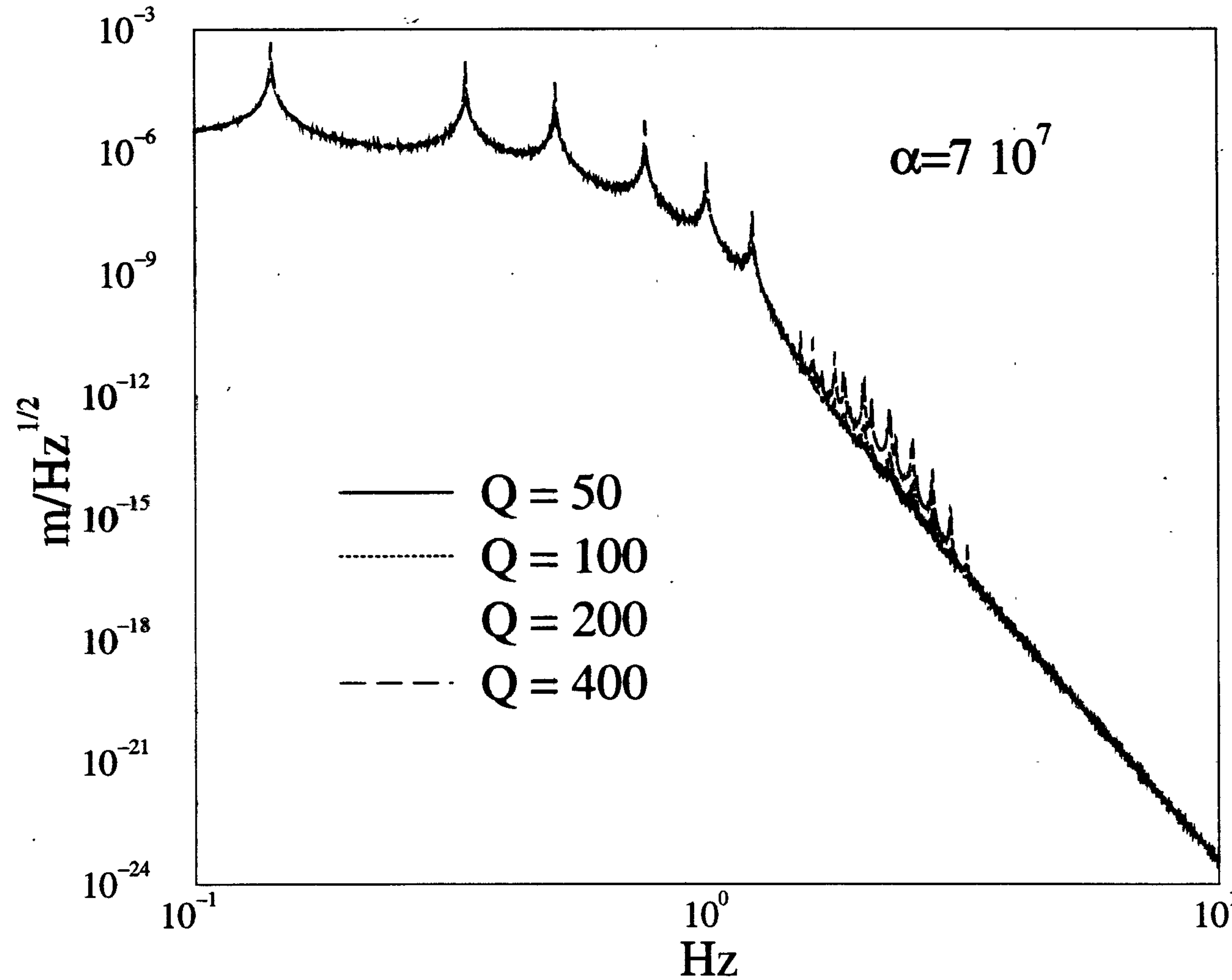


$$k_{antispring} \Delta z + \alpha (\Delta z)^3$$

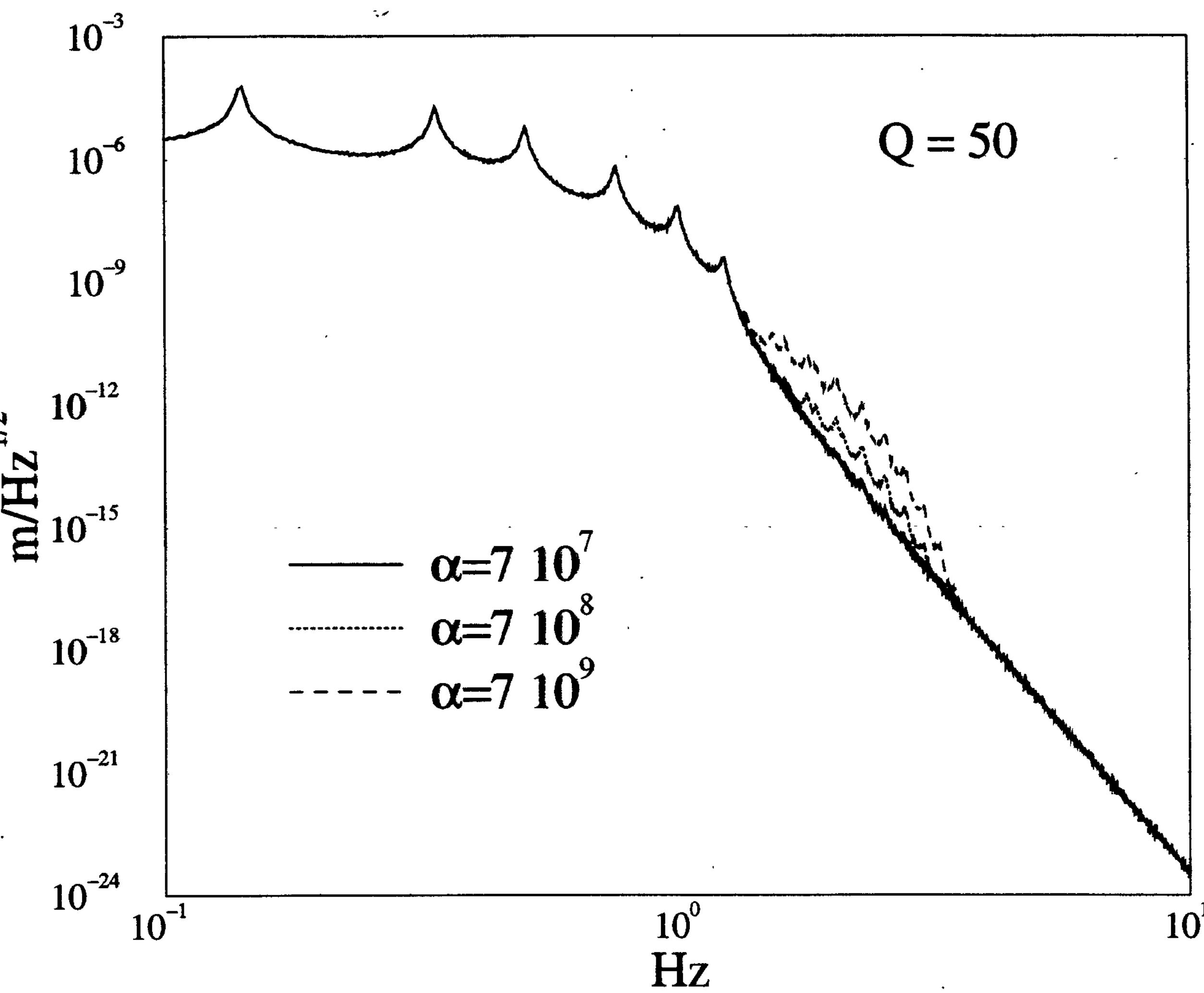
$$k_{antipring} \simeq -1.56\, 10^4 \frac{N}{m}$$

$$\alpha \simeq 4.35\, 10^7 \frac{N}{m^3}$$

For more details "Effects of non-linearities in SA ..."
VIR-NOT-PIS-1390-129



10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
10^{-18}
10^{-21}
10^{-24}
10^{-1}
10^{0}
10^{1}
Hz
m/Hz$^{1/2}$
α=7 10^7
Q = 50
Q = 100
Q = 200
Q = 400



Q = 50
m/Hz$^{1/2}$
Hz
10^{-3}
10^{-6}
10^{-9}
10^{-12}
10^{-15}
10^{-18}
10^{-21}
10^{-24}
10^{-1}
10^{0}
10^{1}
α=7 10^7
α=7 10^8
α=7 10^9

Simple models whose PSD shows a typical low frequency tail

Motivation

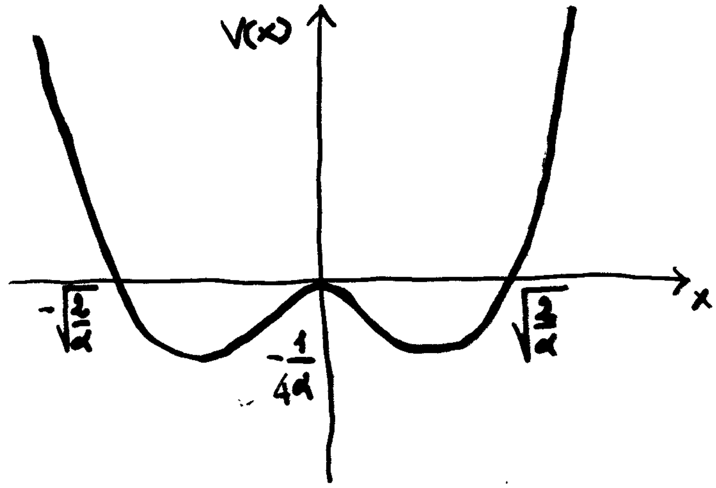
- The overall goal is to investigate how the cooling of many of the normal modes in a mirror influences the thermal spectrum al low frequency.
- The problem is interesting by itself because a lot of systems seem to show the typical tail $\frac{1}{f}$ at low frequency.

Tools

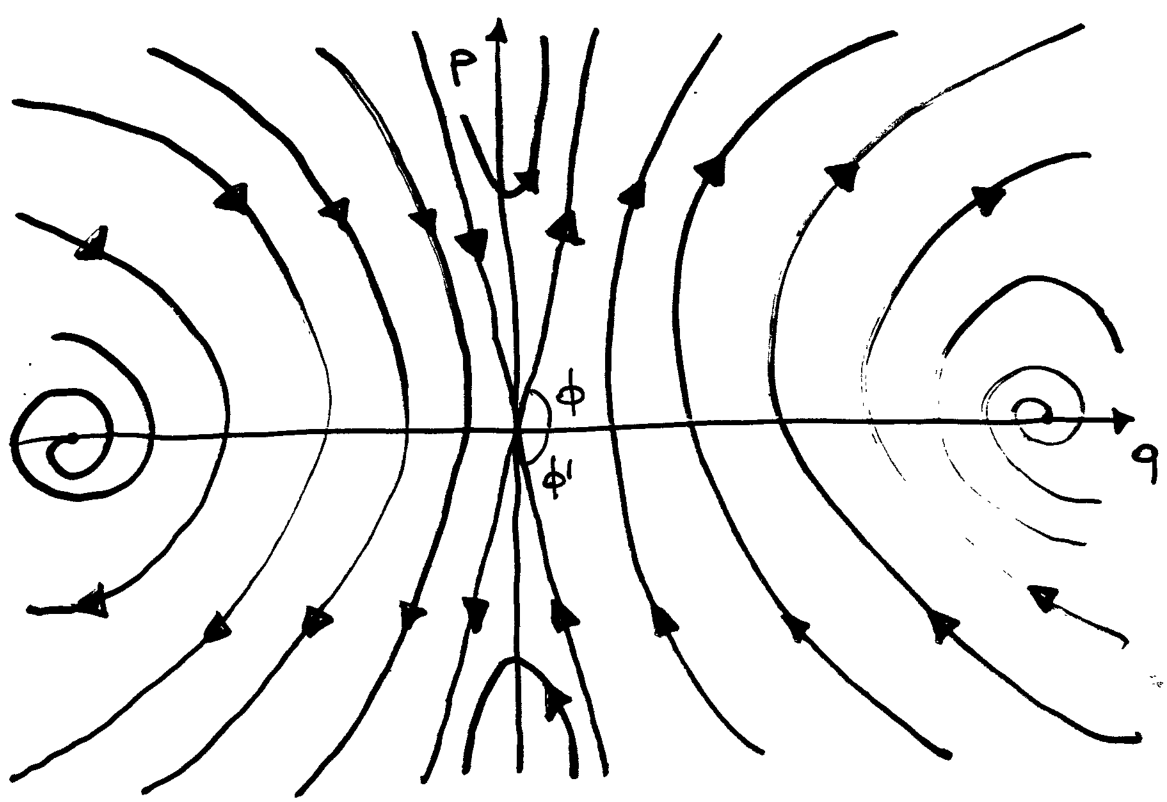
- Critical Phenomena Theory
- Simulation of simple models with a threshold characterization

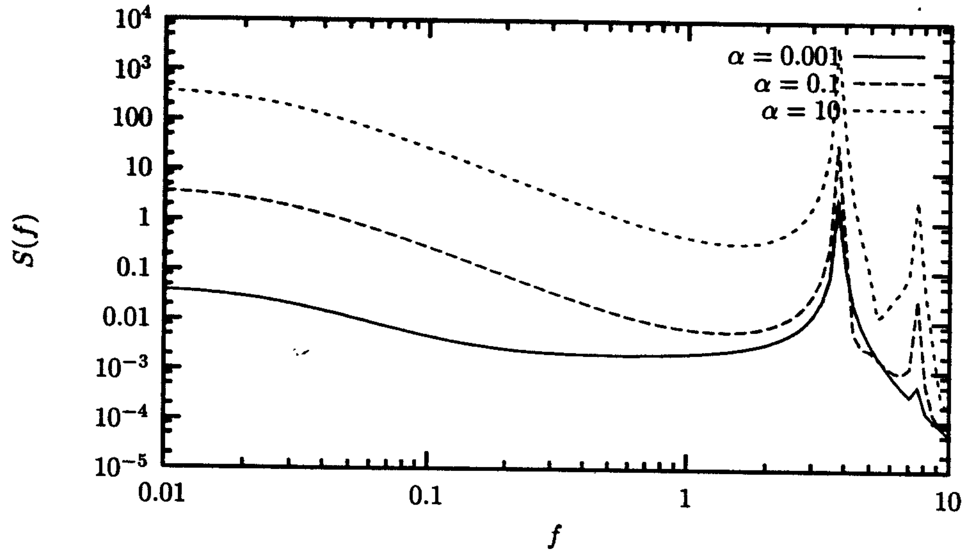
Results

- For the double hole potential a low frequency tail is displayed by simulations.
- A damping force of viscous type doesn't reduce the PSD off the resonance range.

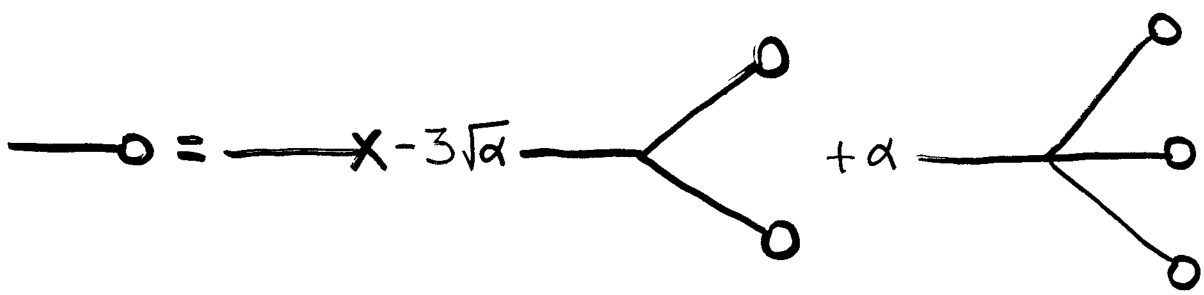


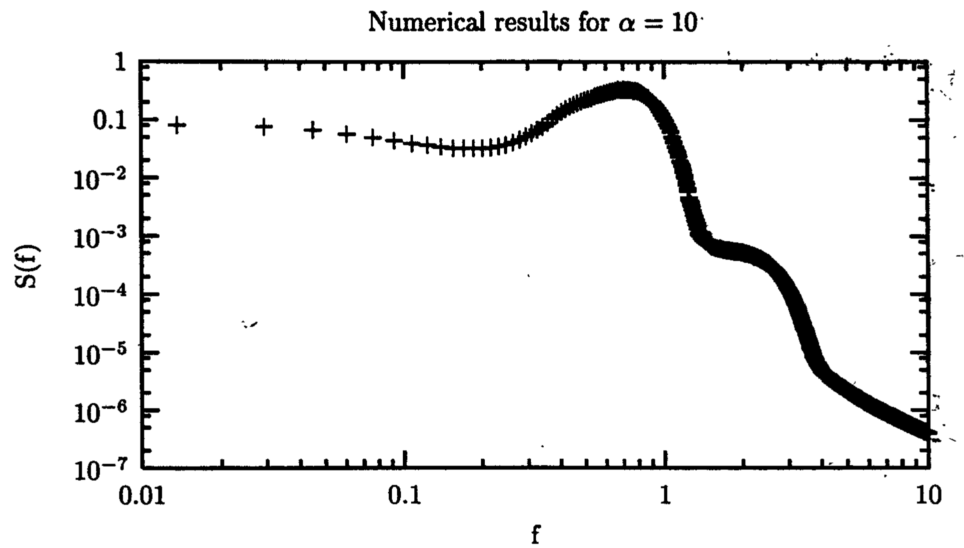
$$V(x) \sim -\frac{x^2}{2} + \alpha\frac{x^4}{4}$$

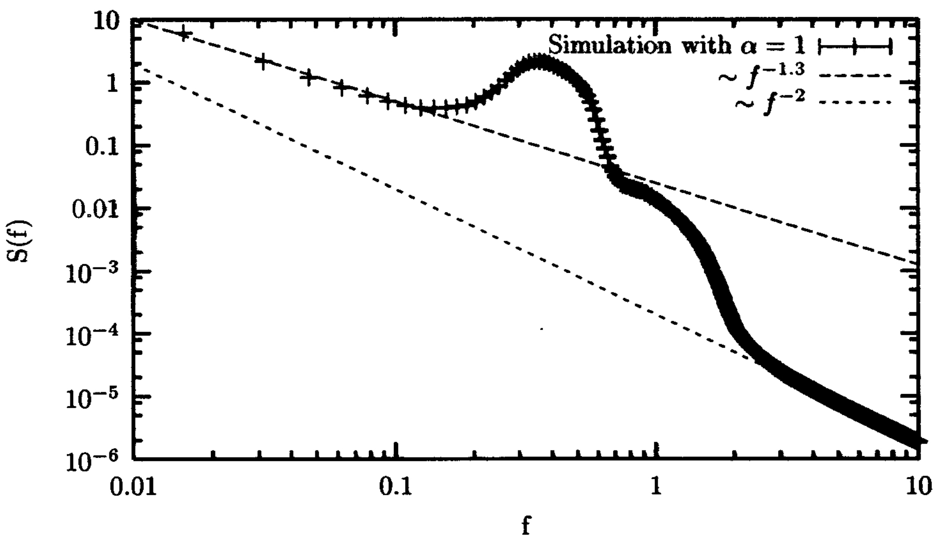


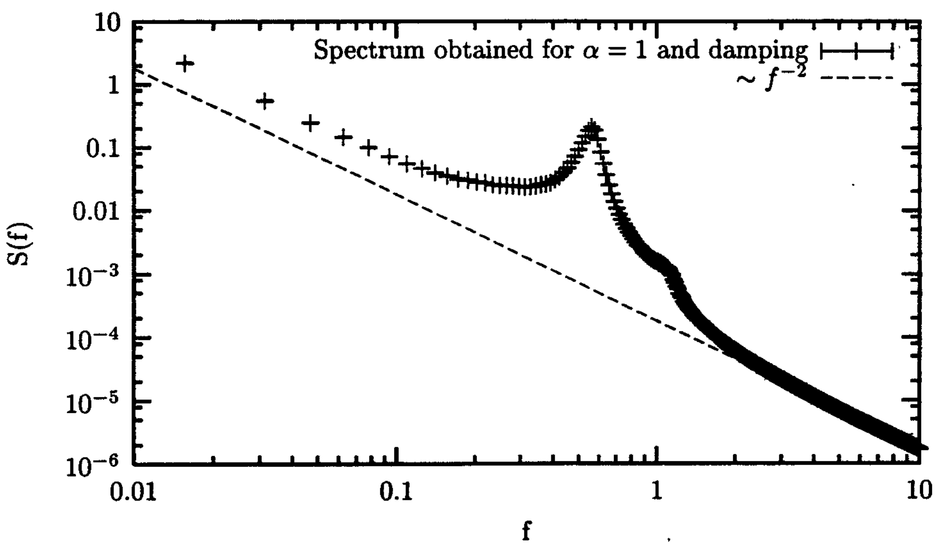


Implicit definition of the exact solution




Numerical results for α = 10
S(f)
f


Simulation with α = 1
~ f^{-1.3}
~ f^{-2}
S(f)
f


Spectrum obtained for α = 1 and damping
~ f^{-2}
S(f)
f