The (multi-IFO) $\mathcal{F}$-statistic metric

LSC-meeting, Hanford, March 2006

LIGO-G060124-00-Z

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Motivation

Coherent search for neutron-star signals with *unknown*
$\mathcal{A} = \{$ amplitude, polarization, orientation, initial phase $\}$
$\lambda = \{$frequency, spindowns, sky-position, (+ binary params)$\}$

$\Rightarrow$ Need optimal covering of (huge) parameter-space

Owen96: template placement based on local *metric*

JKS98: explicit maximization of detection-statistic over "amplitude-parameters" $\mathcal{A} \quad \Longrightarrow \quad \mathcal{F}$-statistic $\mathcal{F}(\lambda; x(t))$

Multi-IFO pulsar signal

multi-IFO data $\{\boldsymbol{x}\}^{\mathrm{X}} = x^{\mathrm{X}}$, with IFO-index X

"data = noise + signal": $\boldsymbol{x}(t) = \boldsymbol{n}(t) + \boldsymbol{s}(t)$

neutron-star signals: $$\boldsymbol{s}(t; \mathcal{A}, \lambda) = \sum_{\mu=1}^{4} \mathcal{A}^\mu \, \boldsymbol{h}_\mu(t; \lambda)$$

$\mathcal{A}^\mu = \mathcal{A}^\mu(h_0, \cos\iota, \psi, \phi_0)$ ... 4 "amplitude parameters"

$\lambda = \{\vec{n}, f, \dot{f}, \ddot{f}, \ldots\}$ ... "Doppler parameters"

$\Rightarrow$ NS parameter-space: $\boxed{\theta = \{\mathcal{A}^\mu, \lambda^i\}}$

Multi-IFO $\mathcal{F}$-statistic

Scalar product: $(\boldsymbol{x}|\boldsymbol{y}) \equiv \int_{-\infty}^{\infty} \widetilde{x}^{\mathrm{X}}(f)\, S^{-1}_{\mathrm{XY}}(f)\, \widetilde{y}^{\mathrm{Y}*}(f)\, df$ (CS05)

Likelihood function: (Gaussian stationary noise)

$$P(\boldsymbol{x}|\mathcal{A}, \lambda, S^{\mathrm{XY}}) = k e^{-\frac{1}{2}(\boldsymbol{n}|\boldsymbol{n})} = k e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{s}|\boldsymbol{x}-\boldsymbol{s})}$$

quadratic in the amplitudes $\mathcal{A}^\mu$ $\Longrightarrow$ maximize over $\mathcal{A}^\mu$

If data $\boldsymbol{x}$ contains a signal with params $\theta_{\mathrm{s}} = \{\mathcal{A}_{\mathrm{s}}, \lambda_{\mathrm{s}}\}$:

$$2\mathcal{F}(\theta_{\mathrm{s}}; \lambda) = x_\mu \mathcal{M}^{\mu\nu} x_\nu$$

where $x_\mu(\theta_{\mathrm{s}}; \lambda) \equiv (\boldsymbol{x}(\theta_{\mathrm{s}})|\boldsymbol{h}_\mu(\lambda))$

and $\mathcal{M}_{\mu\nu}(\lambda) \equiv (\boldsymbol{h}_\mu(\lambda)|\boldsymbol{h}_\nu(\lambda))$

$\mathcal{F}$-metric in λ-space

$E[2\mathcal{F}] = 4 + \mathrm{SNR}^2$, offset: $\lambda = \lambda_\mathrm{s} + \Delta\lambda$

perfect match ($\Delta\lambda = 0$): $\mathrm{SNR}^2(0) = \mathcal{A}_\mathrm{s}{}^\mu \, \mathcal{M}_{\mu\nu} \, \mathcal{A}_\mathrm{s}{}^\nu$

small offset ($\Delta\lambda \ll 1$):

$\mathrm{SNR}^2(\Delta\lambda) = \mathrm{SNR}^2(0) - (\mathcal{A}_\mathrm{s}{}^\mu \, \mathcal{G}_{\mu\nu\,ij} \, \mathcal{A}_\mathrm{s}{}^\nu) \; \Delta\lambda^i \, \Delta\lambda^j + \mathcal{O}(\Delta\lambda^3)$

where $\mathcal{G}_{\mu\nu\,ij}(\lambda) \equiv (\partial_i \boldsymbol{h}_\mu | \partial_j \boldsymbol{h}_\nu) - (\boldsymbol{h}_\alpha | \partial_i \boldsymbol{h}_\mu) \; \mathcal{M}^{\alpha\beta} \; (\boldsymbol{h}_\beta | \partial_j \boldsymbol{h}_\nu)$

$$m_\mathcal{F} \equiv \frac{\mathrm{SNR}^2(0) - \mathrm{SNR}^2(\Delta\lambda)}{\mathrm{SNR}^2(0)} = g^\mathcal{F}_{ij}(\mathcal{A}_\mathrm{s}; \lambda_\mathrm{s}) \, \Delta\lambda^i \, \Delta\lambda^j$$

metric *family*:

$$g^\mathcal{F}_{ij}(\cos\iota, \psi; \lambda_\mathrm{s}) = \frac{\mathcal{A}_\mathrm{s} \cdot \mathcal{G}_{ij}(\lambda_\mathrm{s}) \cdot \mathcal{A}_\mathrm{s}}{\mathcal{A}_\mathrm{s} \cdot \mathcal{M} \cdot \mathcal{A}_\mathrm{s}}$$

The $\mathcal{F}$-metric family

Eliminate dependency on *unknown* amplitudes $\mathcal{A}_{\rm s}$:

Extrema of $m_{\mathcal{F}}(\mathcal{A}_{\rm s}, \lambda_{\rm s}; \Delta\lambda)$ as function of $\mathcal{A}_{\rm s}$: $\frac{\partial m_{\mathcal{F}}}{\partial \mathcal{A}_{\rm s}} = 0$

$\Rightarrow$ eigenvalue problem: $(\mathcal{M}^{-1}\mathcal{G})\,\mathcal{A} = \widehat{m}_{\mathcal{F}}(\lambda, \Delta\lambda)\,\mathcal{A}$

extrema: $m_{\mathcal{F}}(\cos\iota, \psi; \lambda, \Delta\lambda) \in \left[\widehat{m}_{\mathcal{F}}^{\min}(\lambda, \Delta\lambda), \widehat{m}_{\mathcal{F}}^{\max}(\lambda, \Delta\lambda)\right]$

“average”: $\overline{m}_{\mathcal{F}}(\lambda, \Delta\lambda) = \frac{1}{4}\mathrm{Tr}\left[\mathcal{M}^{-1}\mathcal{G}\right] = \overline{g}^{\mathcal{F}}_{ij}(\lambda)\,\Delta\lambda^i\Delta\lambda^j$

Uncorrelated noise, narrow-band signals

uncorrelated noises: $S^{\mathrm{XY}} = S^{\mathrm{X}}\,\delta^{\mathrm{XY}}$
+ narrow-band signals:

$$(\boldsymbol{x}|\boldsymbol{y}) = \sum_{\mathrm{X}} S_{\mathrm{X}}^{-1} \int_0^T x^{\mathrm{X}}(t)\, y^{X}(t)\, dt$$

multi-IFO averaging: $\langle Q \rangle_S \equiv \sum_{\mathrm{X}} w_{\mathrm{X}} \langle Q^{\mathrm{X}} \rangle$, where

$\langle Q \rangle \equiv \frac{1}{T}\int_0^T Q(t)\, dt$ $\quad w_{\mathrm{X}} \equiv \frac{S_{\mathrm{X}}^{-1}}{\widehat{\mathcal{S}}}$ $\quad \sum_{\mathrm{X}} w_{\mathrm{X}} = 1$

$$(\boldsymbol{x}|\boldsymbol{y}) = T\,\widehat{\mathcal{S}}\,\langle x\, y \rangle_S$$

Explicit calculation of $\mathcal{F}$-metric

$$\mathcal{M}_{\mu\nu} \approx \frac{1}{2} T \widehat{\mathcal{S}} \begin{pmatrix} A & C & 0 & 0 \\ C & B & 0 & 0 \\ 0 & 0 & A & C \\ 0 & 0 & C & B \end{pmatrix}$$

$$\mathcal{G}_{\mu\nu\,ij} \approx \frac{1}{2} T \widehat{\mathcal{S}} \begin{pmatrix} m^1_{ij} & m^3_{ij} & 0 & 0 \\ m^3_{ij} & m^2_{ij} & 0 & 0 \\ 0 & 0 & m^1_{ij} & m^3_{ij} \\ 0 & 0 & m^3_{ij} & m^2_{ij} \end{pmatrix}$$

e.g. $m^1_{ij} = \langle a^2\, \partial_i \phi \partial_j \phi \rangle_S - \frac{A}{D} \langle a\, b \partial_i \phi \rangle_S \langle a\, b\, \partial_j \phi \rangle_S + \dots$

recall $g^{\mathcal{F}}_{ij} = \frac{\mathcal{A}\cdot\mathcal{G}_{ij}\cdot\mathcal{A}}{\mathcal{A}\cdot\mathcal{M}\cdot\mathcal{A}}$

Long-duration limit: orbital metric

phase: $\phi^{\mathrm{X}}(t;\lambda) = \phi_{\mathrm{orb}}(t;\lambda) + \Delta\phi^{\mathrm{X}}(t;\lambda)$

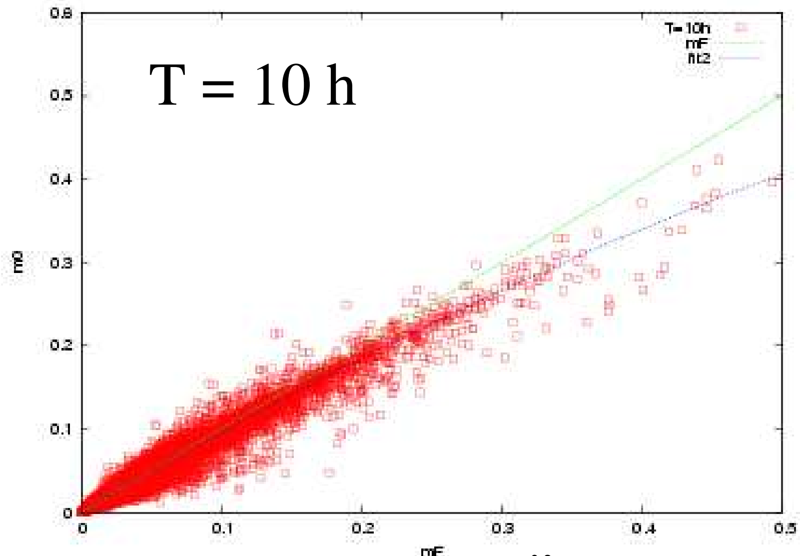
for $T \gg 1$ day:

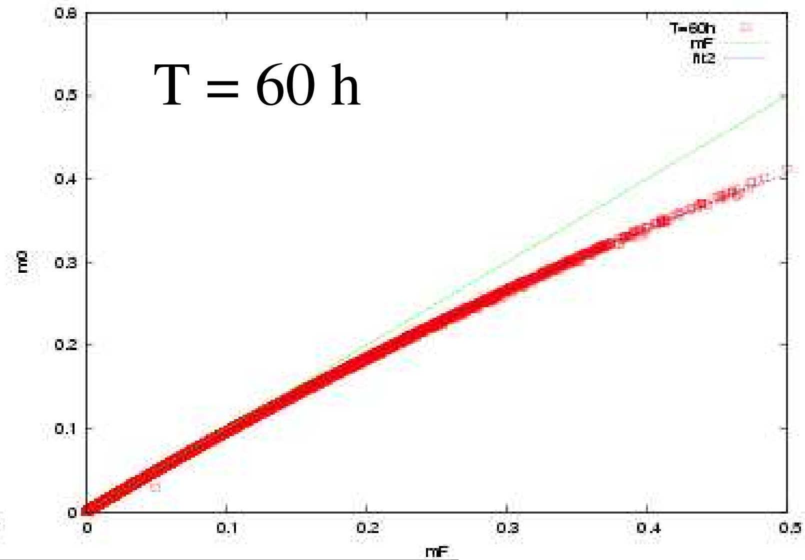
$$g^{\mathcal{F}}_{ij} \longrightarrow g^{\mathrm{orb}}_{ij} \equiv \langle \partial_i \phi_{\mathrm{orb}} \partial_j \phi_{\mathrm{orb}} \rangle - \langle \partial_i \phi_{\mathrm{orb}} \rangle \langle \partial_j \phi_{\mathrm{orb}} \rangle$$

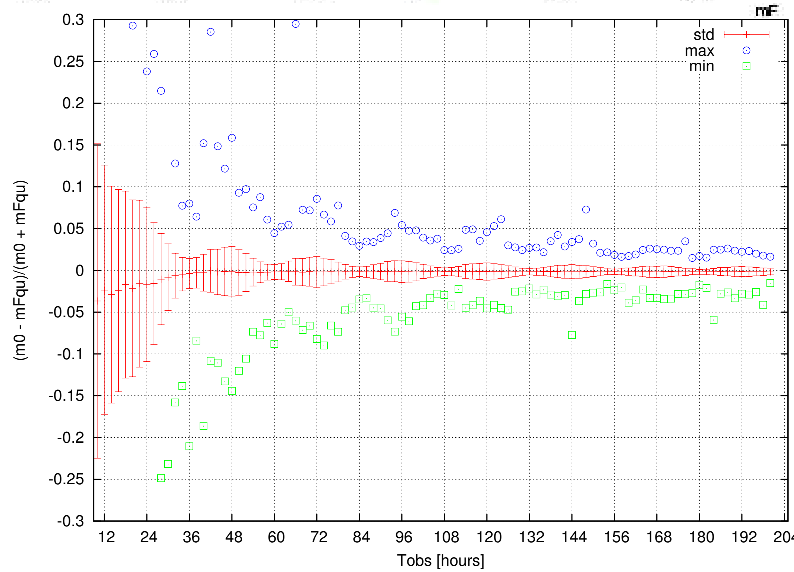
ϕ_{orb} is independent of detector!

Comparison to measured mismatch

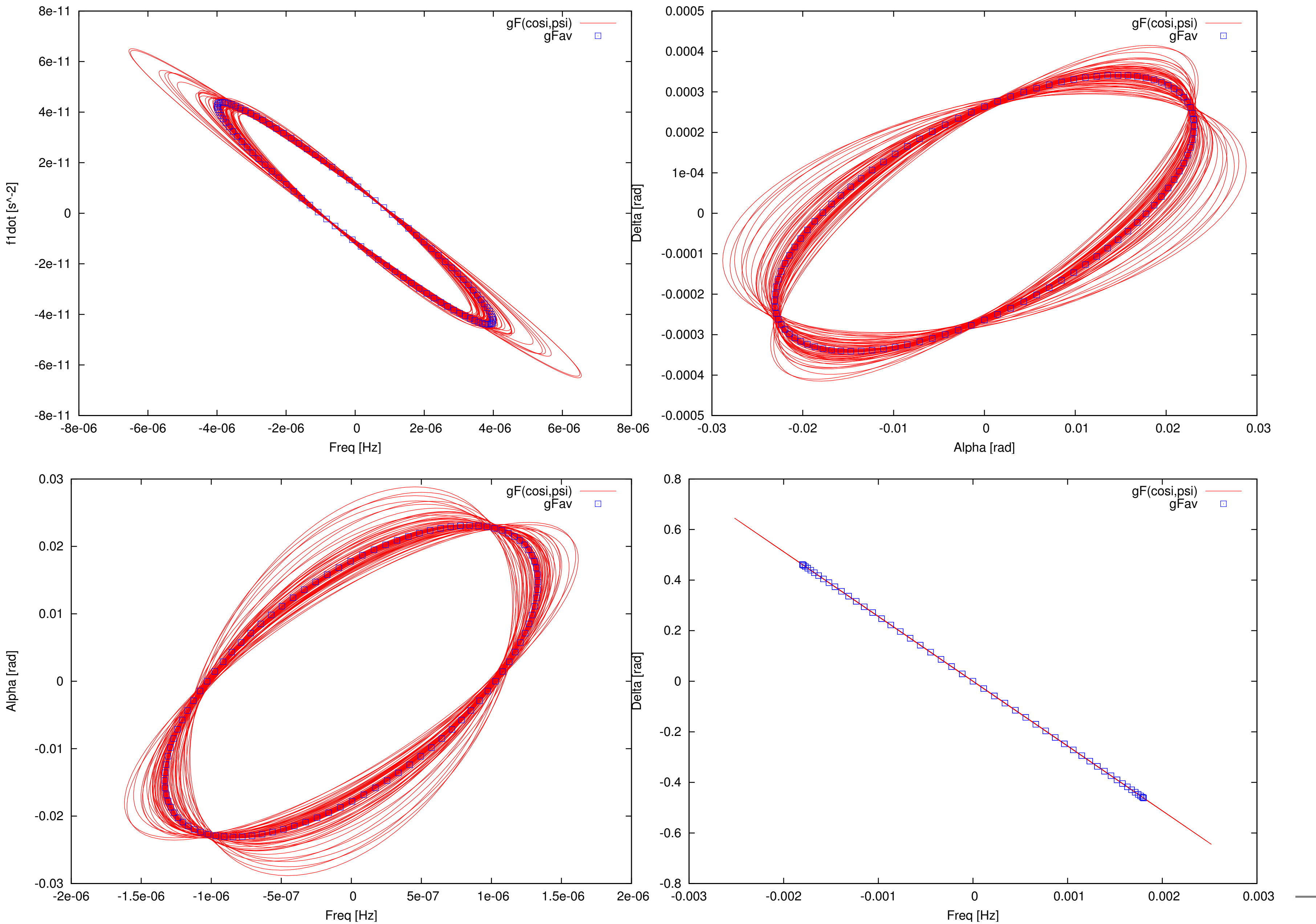
Predicted $m_{\mathcal{F}}(\mathcal{A}_{\rm s}, \lambda_{\rm s}; \Delta\lambda)$ versus *measured* m_0:



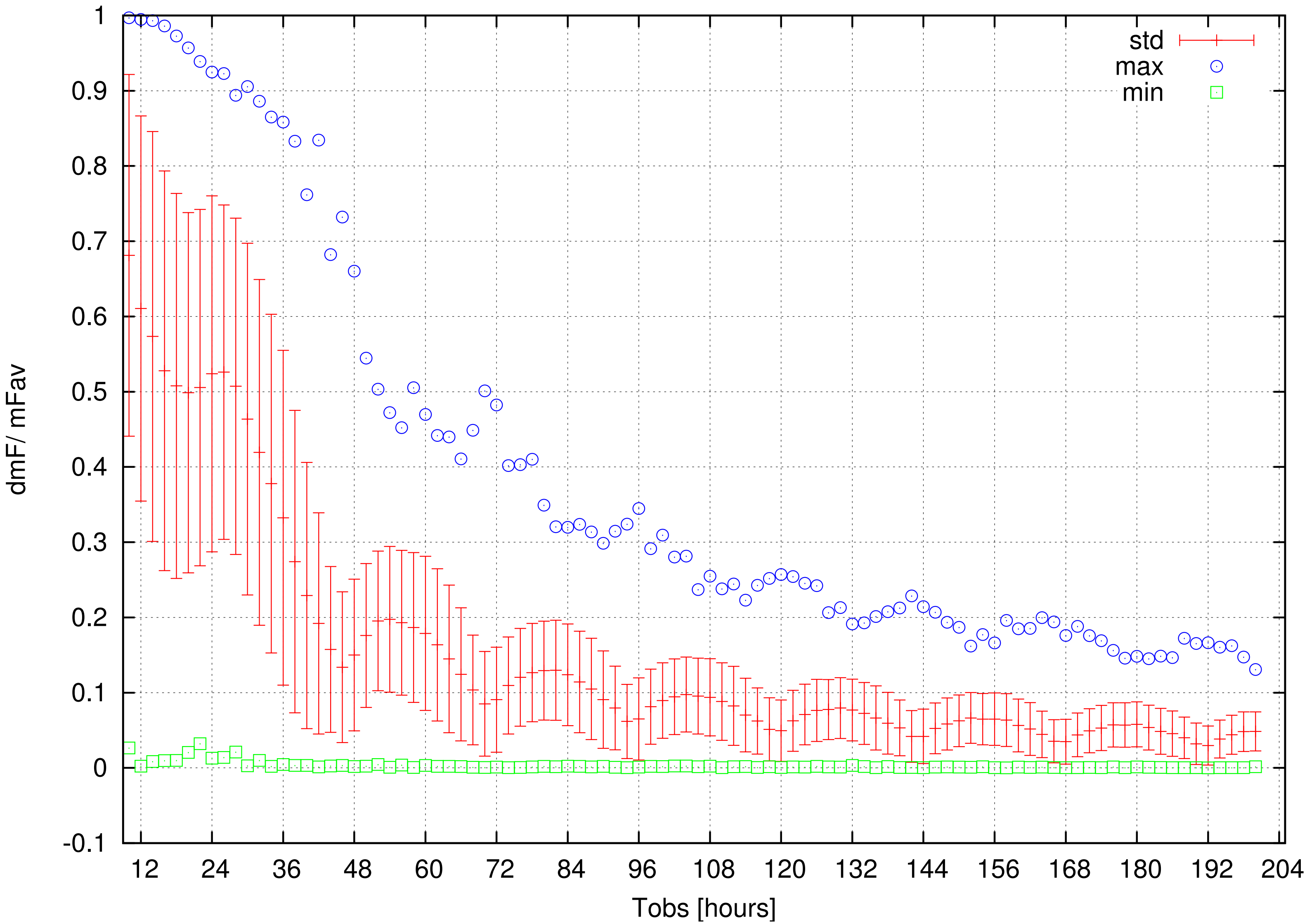




The $\mathcal{F}$-metric family $(T = 50h)$

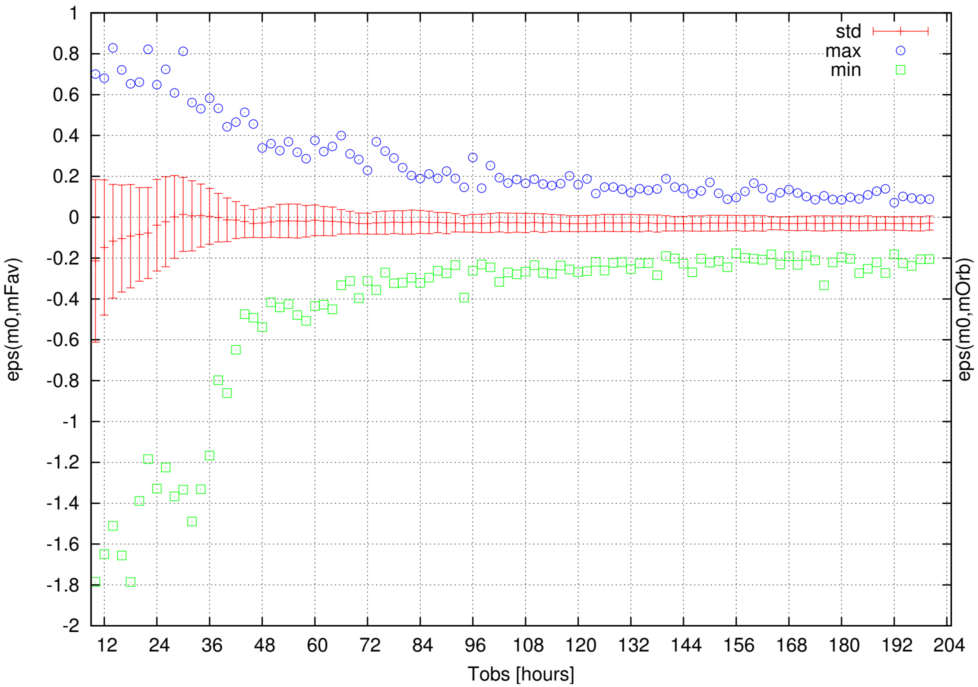


Intrinsic uncertainty of $\mathcal{F}$-metric

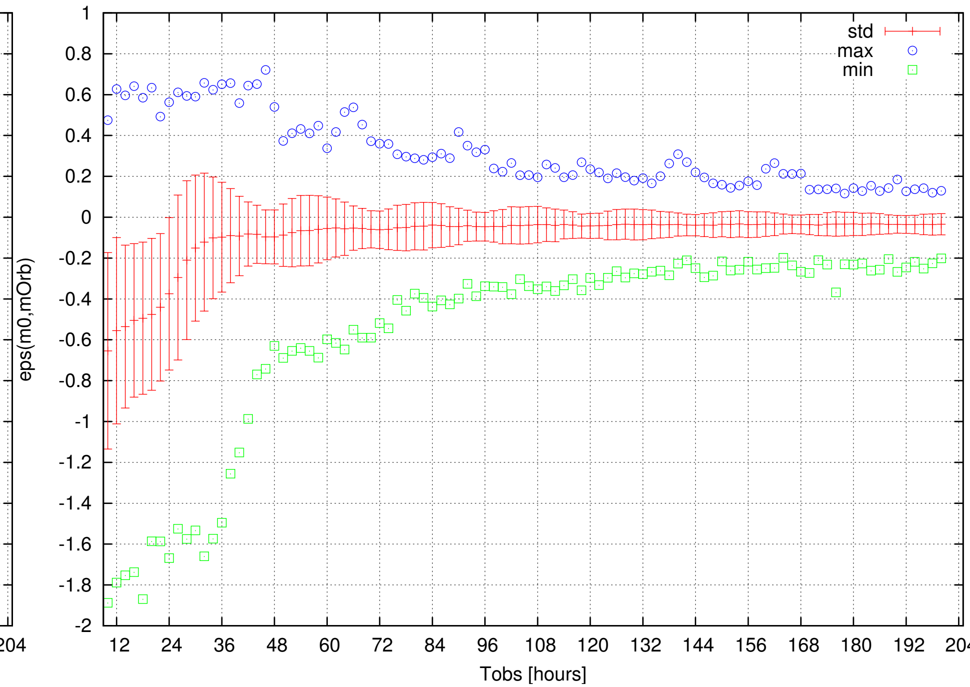


Quality of average and orbital metric

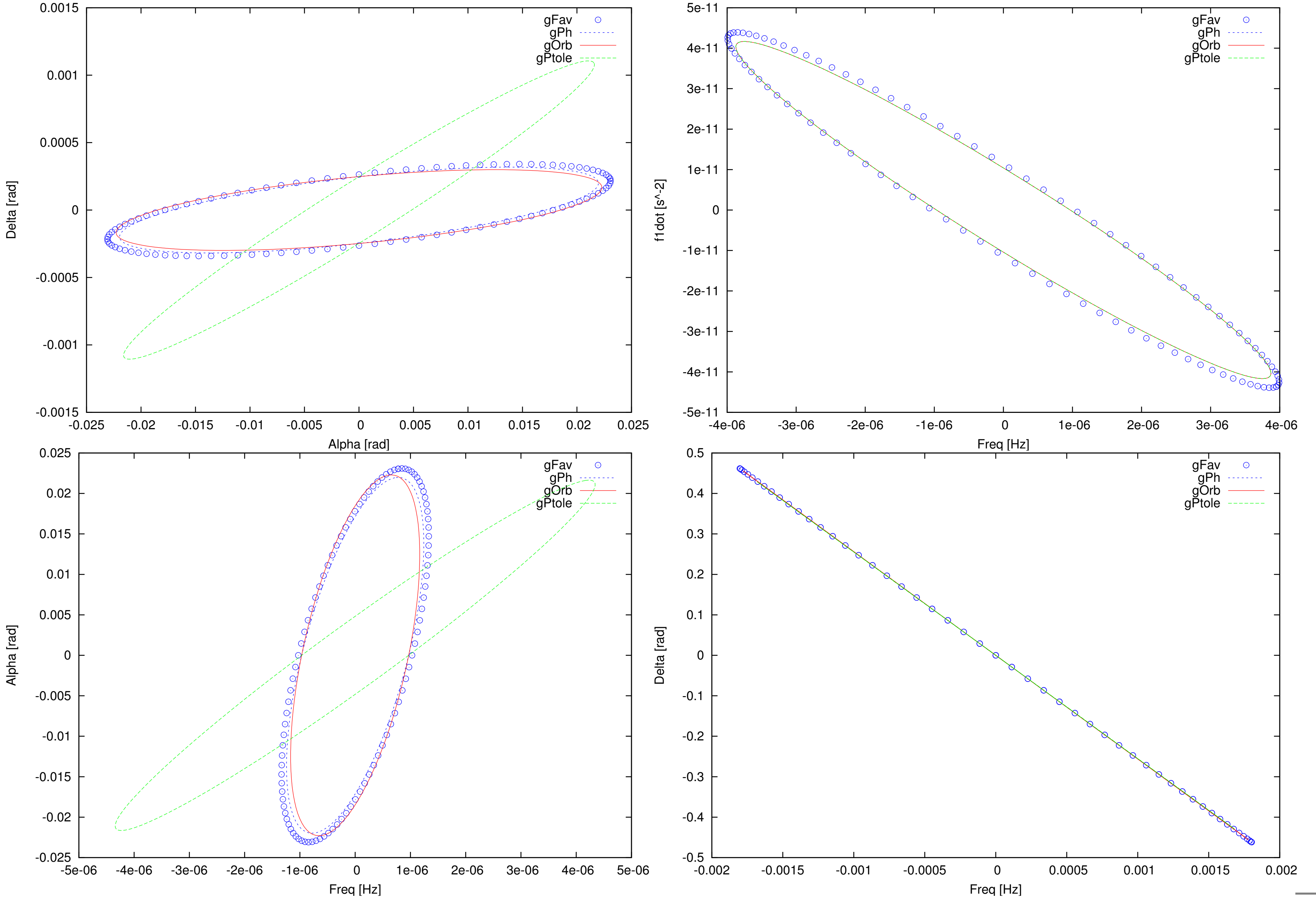
$\bar{g}^{\mathcal{F}}_{ij}$



g^{orb}_{ij}

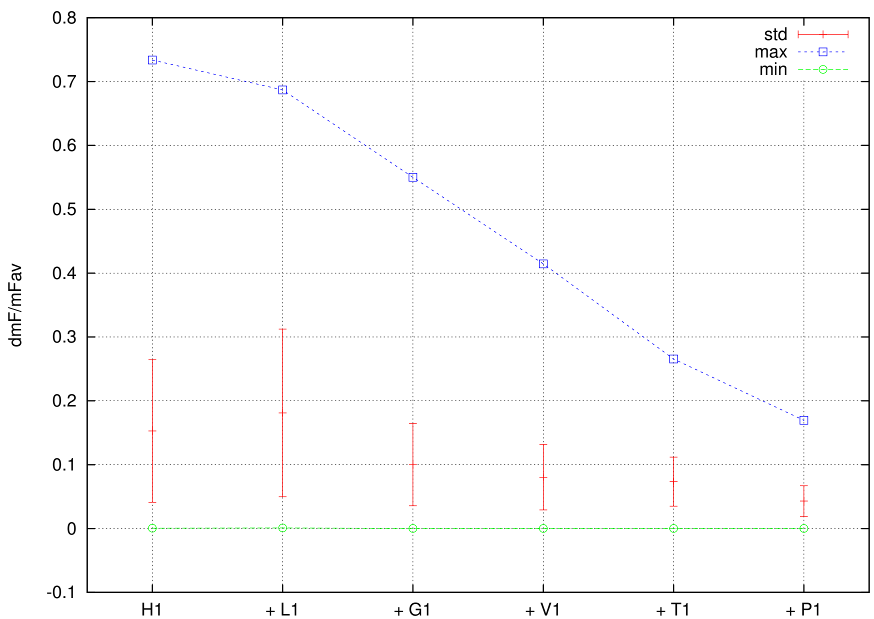


Different metric approximations

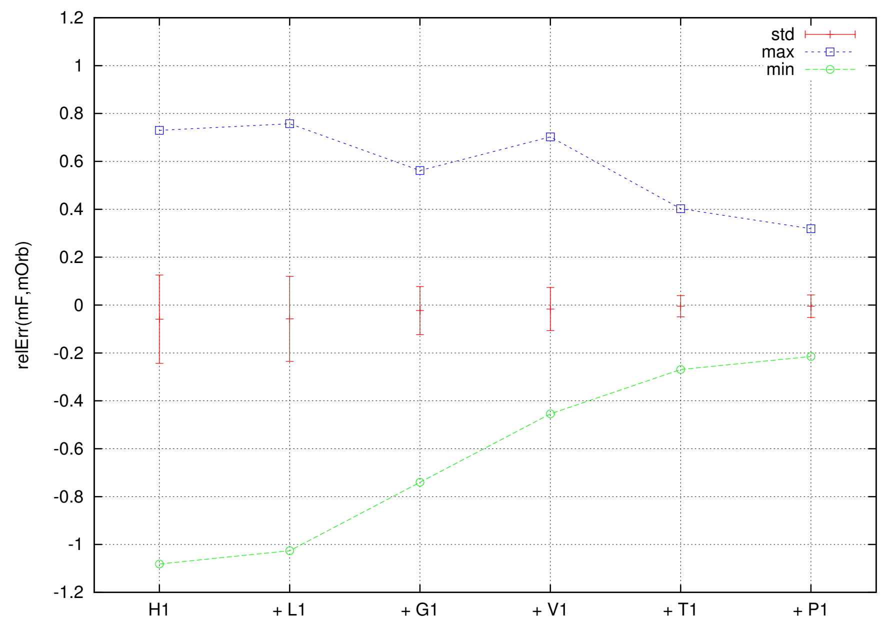


Dependence on number of detectors

Uncertainty $\Delta m_{\mathcal{F}} / \overline{m}_{\mathcal{F}}$



Relative difference $\varepsilon(m_{\mathcal{F}}, m_{\mathrm{orb}})$



Main results

- Generally: no single $\mathcal{F}$-metric, but *family* with unknown parameters $\{\cos\iota, \psi\}$ $\Longrightarrow$ "intrinsic uncertainty"
- uncertainty converges to *zero* with increasing observation-time (T ~days), and also decreases with number of detectors
- long-duration limiting metric is the *orbital phase metric*, which is *independent* of detectors (and *flat*!)
- Ptolemaic approximation is less reliable than the orbital metric due to orientation-error of the metric ellipses ($\rightarrow$ orbital velocity vector off by $\sim 1 - 3^\circ$)