USING GENERALIZED POWERFLUX METHODS TO ESTIMATE THE PARAMETERS OF PERIODIC GRAVITATIONAL WAVES

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1 Parameter Estimation Using Power

A periodic gravitational wave signal incident on a gravitational wave detector produces a strain response of the form,

$$h(t) = A_+ F_+(\psi, t) \cos \Phi(t) + A_\times F_\times(\psi, t) \sin \Phi(t), \tag{1}$$

where F_+ and $F_\times$ are the antenna-pattern response functions, and Φ is the phase [1].

For a time-baseline ≤ 30 minutes we call the Discrete Fourier Transform (DFT) of the data a Short Fourier Transform (SFT). By using the power in SFTs the PowerFlux method [2] is able to estimate the square amplitude of a linearly polarized ($A_+^2 = 0$ or $A_\times^2 = 0$) or circularly polarized signal ($A_+^2 = A_\times^2$). The estimated square amplitude can be used as a detection statistic to search for elliptically polarized signals.

We have tried using the complex SFT amplitudes to estimate A_+ and $A_\times$ based on the methods given in [1]. This failed to be robust, for example because mismatch can cause complete loss of signal in one SFT [3]. Instead, using power to estimate the square amplitudes reduces this loss to at most a 60% in any one SFT. Pursing this, we find several generalization of the PowerFlux method, which we investigate here.

For an SFT, we can treat the frequency and antenna-pattern as constant. Ignoring the mismatch in frequency (which is unknown in a search), the normalized signal power is,

$$\frac{2|\tilde{h}|^2}{T_{\rm SFT}} = 0.5(A_+^2 F_+^2 + A_\times^2 F_\times^2) T_{\rm SFT}, \tag{2}$$

where $\tilde{h}$ is the DFT of $h(t)$ multiplied by one over the sample rate. From here on it is understood that power is to be measured in the SFT bin closest to the signal frequency, and that F_+ and $F_\times$ are constants evaluated at the midpoint of each SFT.

2 Derivation of the PowerFlux method

Equation (2) represents the expected signal power for an elliptically polarized signal from one SFT. If we label the SFTs using index α, and consider a linearly polarized signal with $A_\times = 0$, we can define the noise weighed sum of the square deviations in power as

$$g = \sum_\alpha \frac{[P_\alpha - 0.5 A_+^2 F_{+\alpha}^2 T_{\rm SFT}]^2}{S_\alpha^2}, \tag{3}$$

where S_α is the one-sided power spectral density for the appropriate frequency bin, and

$$P_\alpha = \frac{2|\tilde{x}_\alpha|^2}{T_{\rm SFT}}, \tag{4}$$

with $\tilde{x}_\alpha$ the SFT data from the same frequency bin.

A natural way one way to estimate A_+^2, analogous to χ^2 minimization, is to find the value that minimizes g. Thus, we need to solve

$$\frac{\partial g}{\partial A_+^2} = -\sum_\alpha \frac{(P_\alpha - 0.5A_+^2 F_{+\alpha}^2 T_{\rm SFT})F_{+\alpha}^2 T_{\rm SFT}}{S_\alpha^2} = 0. \tag{5}$$

Solving for A_+^2 gives,

$$A_+^2 = 4\sum_\alpha \frac{F_{+\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2} \Big/ \sum_\alpha \frac{F_{+\alpha}^4}{S_\alpha^2}. \tag{6}$$

Note that equation (6) is the detection statistic of the PowerFlux method defined in [2], though the derivation here is different. We call this the PowerFlux Linear (polarization) method. This search includes a search over values of ψ. It is trivial to generalize this to circular polarization, by replacing $F_{+\alpha}^2$ with $F_{+\alpha}^2 + F_{\times\alpha}^2$ in Eq. (3), giving the PowerFlux Circular (polarization) method, which does not require a search over ψ.

3 Generalization to estimate A_+^2 and $A_\times^2$

The obvious generalization of Eq. (3) to an elliptically polarized signal is to redefine g as

$$g = \sum_\alpha \frac{[P_\alpha - 0.5(A_+^2 F_{+\alpha}^2 + A_\times^2 F_{\times\alpha}^2)T_{\rm SFT}]^2}{S_\alpha^2}. \tag{7}$$

Thus minimizing g with repect to A_+^2 and $A_\times^2$ gives

$$\frac{\partial g}{\partial A_+^2} = -\sum_\alpha \frac{[P_\alpha - 0.5(A_+^2 F_{+\alpha}^2 + A_\times^2 F_{\times\alpha}^2)T_{\rm SFT}]F_{+\alpha}^2 T_{\rm SFT}}{S_\alpha^2} = 0, \tag{8a}$$

$$\frac{\partial g}{\partial A_\times^2} = -\sum_\alpha \frac{[P_\alpha - 0.5(A_+^2 F_{+\alpha}^2 + A_\times^2 F_{\times\alpha}^2)T_{\rm SFT}]F_{\times\alpha}^2 T_{\rm SFT}}{S_\alpha^2} = 0. \tag{8b}$$

Solving for A_+^2 and $A_\times^2$ gives

$$A_+^2 = \frac{4}{\mathcal{D}}\left[\sum_\alpha \frac{F_{\times\alpha}^4}{S_\alpha^2}\sum_\alpha \frac{F_{+\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2} - \sum_\alpha \frac{F_{+\alpha}^2 F_{\times\alpha}^2}{S_\alpha^2}\sum_\alpha \frac{F_{\times\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2}\right], \tag{9a}$$

$$A_\times^2 = \frac{4}{\mathcal{D}}\left[\sum_\alpha \frac{F_{+\alpha}^4}{S_\alpha^2}\sum_\alpha \frac{F_{\times\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2} - \sum_\alpha \frac{F_{+\alpha}^2 F_{\times\alpha}^2}{S_\alpha^2}\sum_\alpha \frac{F_{+\alpha}^2}{S_\alpha^2}\frac{|\tilde{x}_\alpha|^2}{T_{\rm SFT}^2}\right], \tag{9b}$$

where

$$\mathcal{D} = \sum_\alpha \frac{F_{+\alpha}^4}{S_\alpha^2}\sum_\alpha \frac{F_{\times\alpha}^4}{S_\alpha^2} - \left(\sum_\alpha \frac{F_{+\alpha}^2 F_{\times\alpha}^2}{S_\alpha^2}\right)^2. \tag{9c}$$

One can use the sum of A_+^2 and $A_\times^2$ as the detection statistic. This method still has to include a search over values of ψ. Computationally, it involves computing $5/3$ as many sums as the basic PowerFlux Method. We call this method PowerFlux Generalization I.

4 Generalization to estimate A_+^2, $A_\times^2$, and ψ

We can re-write F_+ and $F_\times$ in terms of ψ and two functions independent of ψ, a and b [1]:

$$F_+(\psi, t) = \sin\zeta[\cos 2\psi\, a(t) + \sin 2\psi\, b(t)], \tag{10a}$$

$$F_\times(\psi, t) = \sin\zeta[\cos 2\psi\, b(t) - \sin 2\psi\, a(t)]. \tag{10b}$$

The normalized signal power can be written as,

$$\frac{2|\tilde{h}_\alpha|^2}{T_{\mathrm{SFT}}} = 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}, \tag{11}$$

where the amplitudes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ are

$$\mathcal{A} = \sin^2\zeta(A_+^2\cos^2 2\psi + A_\times^2\sin^2 2\psi), \tag{12a}$$

$$\mathcal{B} = \sin^2\zeta(A_+^2\sin^2 2\psi + A_\times^2\cos^2 2\psi), \tag{12b}$$

$$\mathcal{C} = \sin^2\zeta(A_+^2 - A_\times^2)2\cos 2\psi\sin 2\psi. \tag{12c}$$

It is easy to invert these equations. Thus, we can redefine g as

$$g = \sum_\alpha \frac{[P_\alpha - 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}]^2}{S_\alpha^2}. \tag{13}$$

Thus minimizing g with respect to A_+^2 and $A_\times^2$ gives

$$\frac{\partial g}{\partial \mathcal{A}} = -\sum_\alpha \frac{[P_\alpha - 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}]a_\alpha^2 T_{\mathrm{SFT}}}{S_\alpha^2} = 0, \tag{14a}$$

$$\frac{\partial g}{\partial \mathcal{B}} = -\sum_\alpha \frac{[P_\alpha - 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}]b_\alpha^2 T_{\mathrm{SFT}}}{S_\alpha^2} = 0, \tag{14b}$$

$$\frac{\partial g}{\partial \mathcal{C}} = -\sum_\alpha \frac{[P_\alpha - 0.5(\mathcal{A}a_\alpha^2 + \mathcal{B}b_\alpha^2 + \mathcal{C}a_\alpha b_\alpha)T_{\mathrm{SFT}}]a_\alpha b_\alpha T_{\mathrm{SFT}}}{S_\alpha^2} = 0. \tag{14c}$$

Thus, the amplitudes $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$ can be found by inverting Eqs. (14). From these the amplitudes A_+^2 and $A_\times^2$ and polarization angle ψ can be found by inverting Eqs. (12). Computationally, this method involves computing $8/3$ as many sums as the basic PowerFlux Method, but because the value for ψ no longer has to be searched over (and in the standard PowerFlux search four linear and one circularly polarization are used) the computational complexity of the search can be less than that for the basic method. We call this method PowerFlux Generalization II.

5 Comparison of Detection Efficiencies

We have written code that generates fake signals and noise, and analyzed it using the above methods plus a pure StackSlide sum of the power (see [4]). The results are given in Figures 1 and 2.

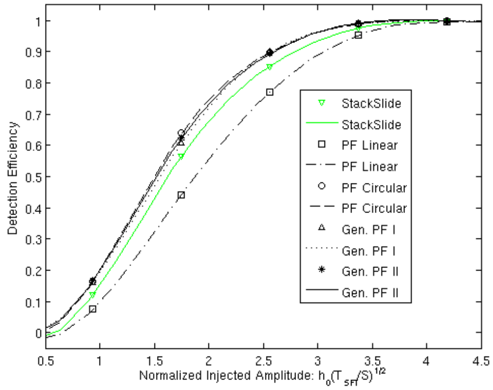

Figure 1: Detection efficiency vs. the normalized injected amplitude, $h_0(T_{\rm SFT}/S)^{1/2}$, for a 1% false alarm rate, where $A_+ = 0.5h_0(1+\cos^2\iota)$ and $A_\times = h_0\cos\iota$, and $\cos\iota$ is the inclination angle. Fake signals were generated for 336 SFTs, equatorial sky position $(0, \pi/4)$, and for random inclination and polarization, allowing for a frequency mismatch of up to one half of an SFT bin, and polarization mismatch of up to $\pi/16$. Spline fits are shown. Errors are $\leq 3\%$. Varying the declination of the sky position, the ranking of StackSlide, Gen. PF I, and Gen. PF II can switch, while PF Circular always remains the most efficient method.

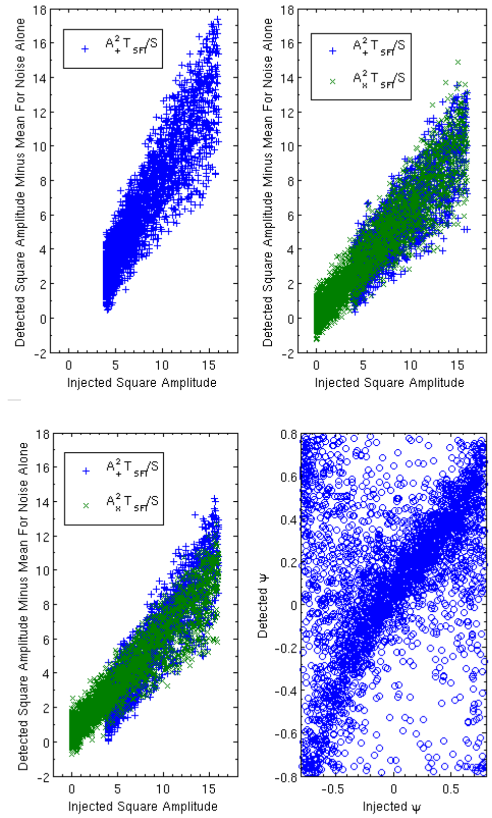

Figure 2: The figures show, for $h_0(T_{\rm SFT}/S)^{1/2} = 4$ as defined in Fig. 1 and 3000 random signals, a comparison of the injected and detected amplitude A_+ for the PowerFlux Linear (polarization) method (top left), a comparison of the injected and detected amplitudes A_+ and $A_\times$ for PowerFlux Generalization I (top right), and a comparison of the injected and detected amplitudes A_+ and $A_\times$ and the injected and detected ψ for PowerFlux Generalization II (bottom left and right).

References

[1] P. Jaranowski, A. Królak, and B. F. Schutz, Phys. Rev. D **58**, 063001 (1998), gr-qc/9804014

[2] V. Dergachev and K. Riles, LIGO-T050186-00-Z

[3] G. Mendell and K. Wette, LIGO-T060286-00-Z

[4] The LIGO Scientific Collaboration, *All-sky LIGO search for periodic gravitational waves in the S4 data*, in preparation.

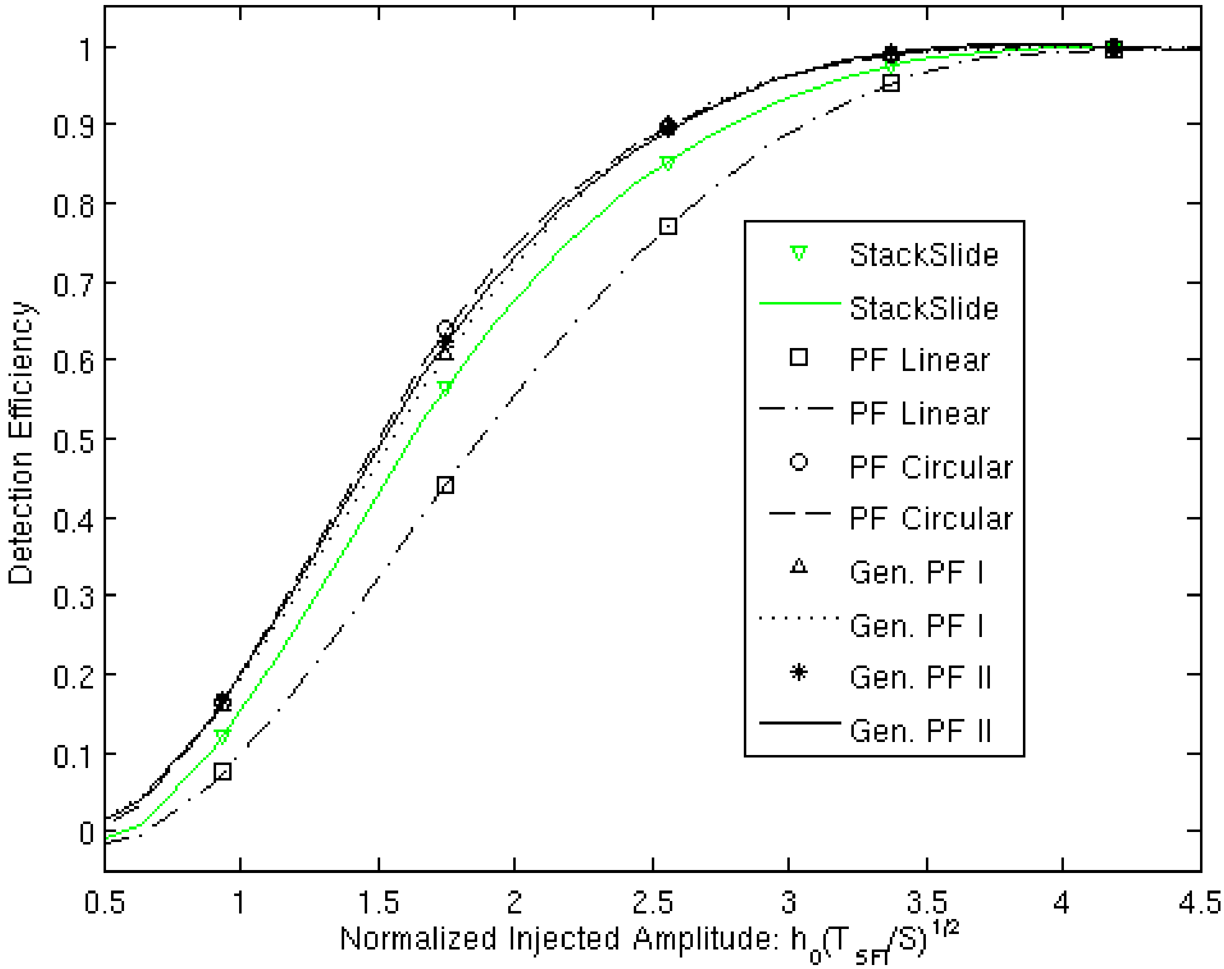

Figure 3: Larger version of Figure 1: detection efficiency vs. the normalized injected amplitude, $h_0(T_{\rm SFT}/S)^{1/2}$, for a 1% false alarm rate, where $A_+ = 0.5h_0(1+\cos^2\iota)$ and $A_\times = h_0\cos\iota$, and $\cos\iota$ is the inclination angle. Fake signals were generated for 336 SFTs, equatorial sky position $(0, \pi/4)$ (Right Ascension = 0 hrs, Declination = 45 degrees), and for random inclination and polarization, allowing for a frequency mismatch of up to one half of an SFT bin, and polarization mismatch of up to $\pi/16$. Spline fits are shown. Errors are $\leq 3\%$. Varying the declination of the sky position, the ranking of StackSlide, Gen. PF I, and Gen. PF II can switch, while PF Circular always remains the most or nearly most efficient method.

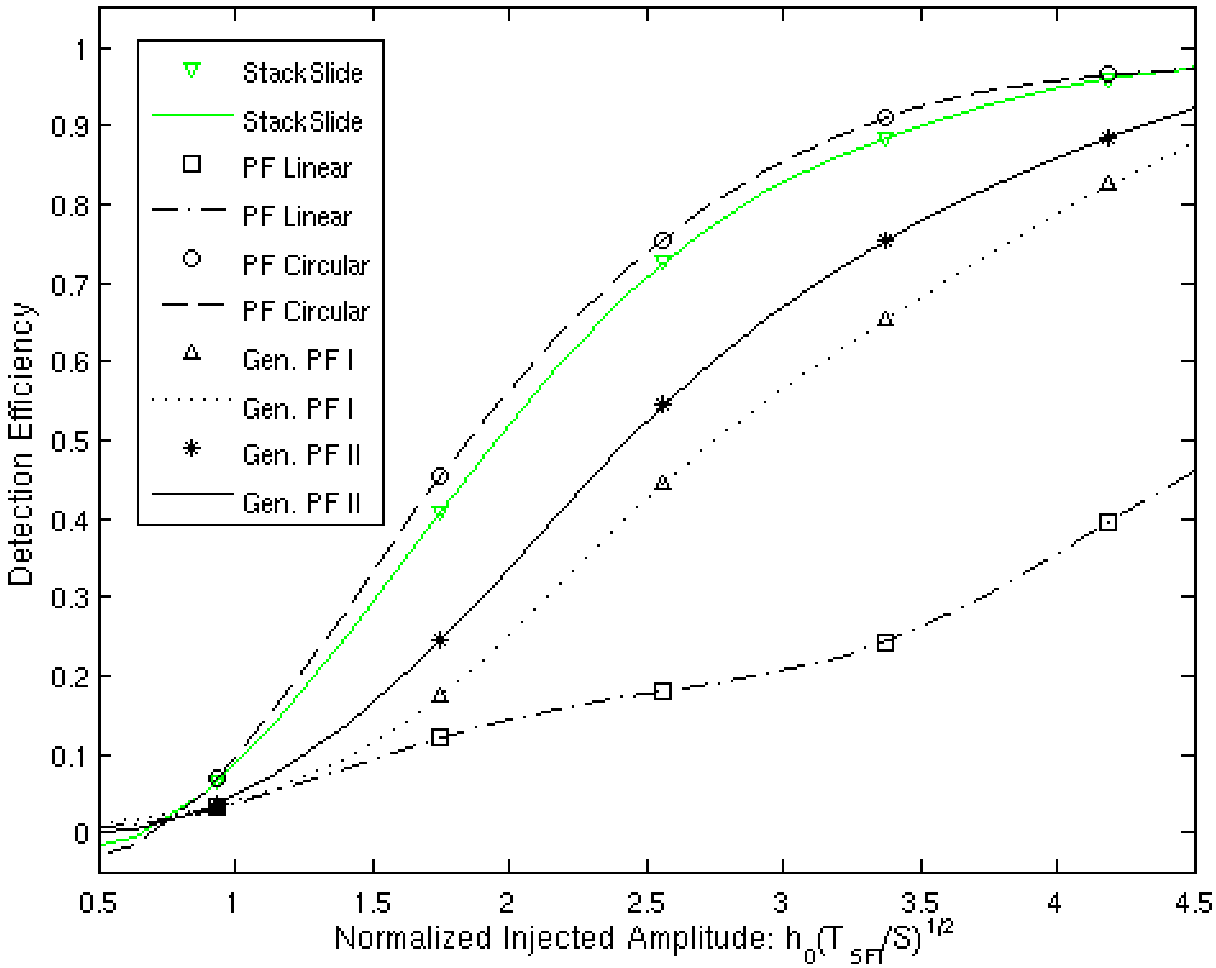

Figure 4: Same as Figure 3 for Declination = 0 degrees.

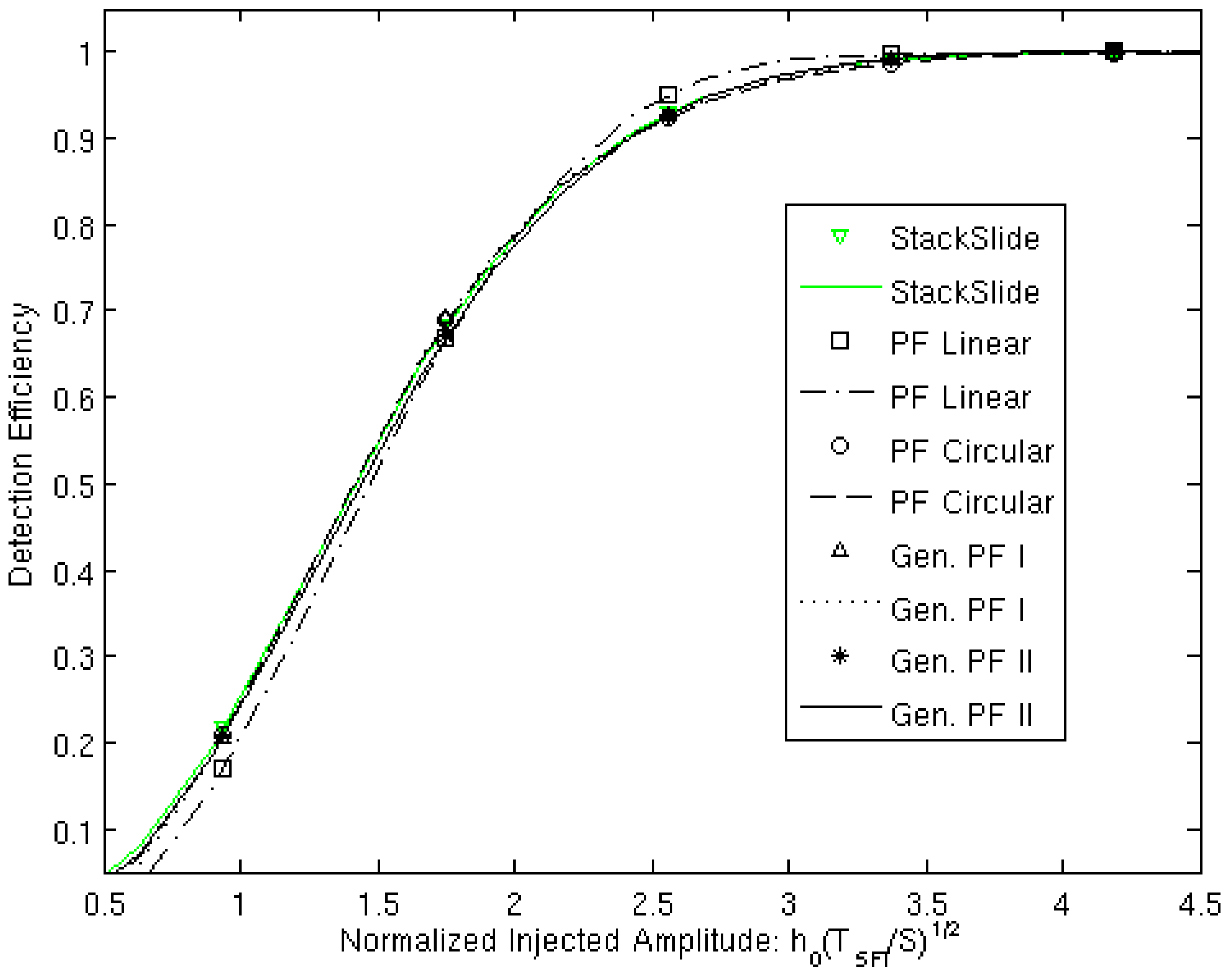

Figure 5: Same as Figure 3 for Declination = 90 degrees.

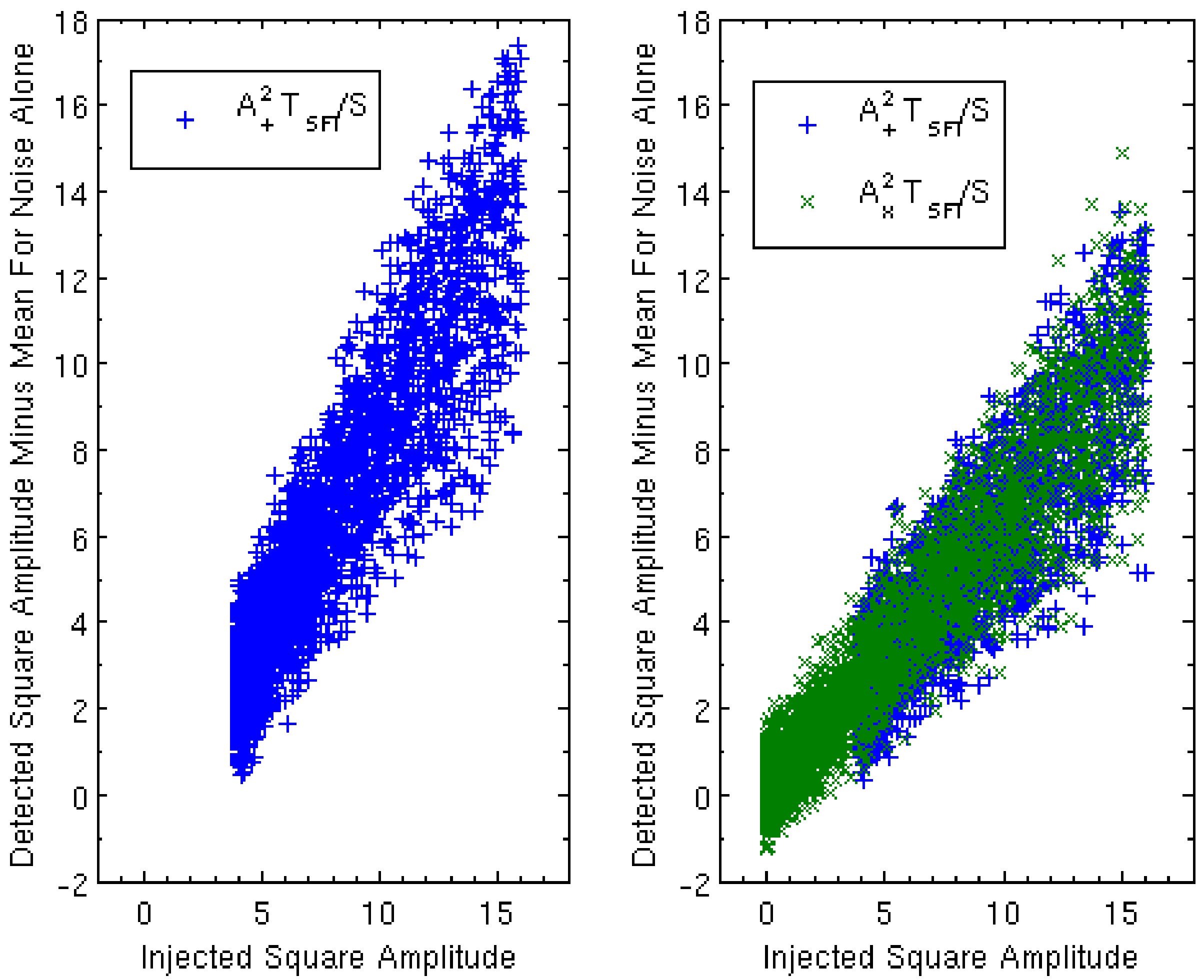

Figure 6: The figures show, for $h_0(T_{\rm SFT}/S)^{1/2} = 4$ and 3000 random signals, a comparison of the injected and detected amplitude A_+^2 for the PowerFlux Linear (polarization) method (left), and a comparison of the injected and detected amplitudes A_+^2 and $A_\times^2$ for PowerFlux Generalization I (right).

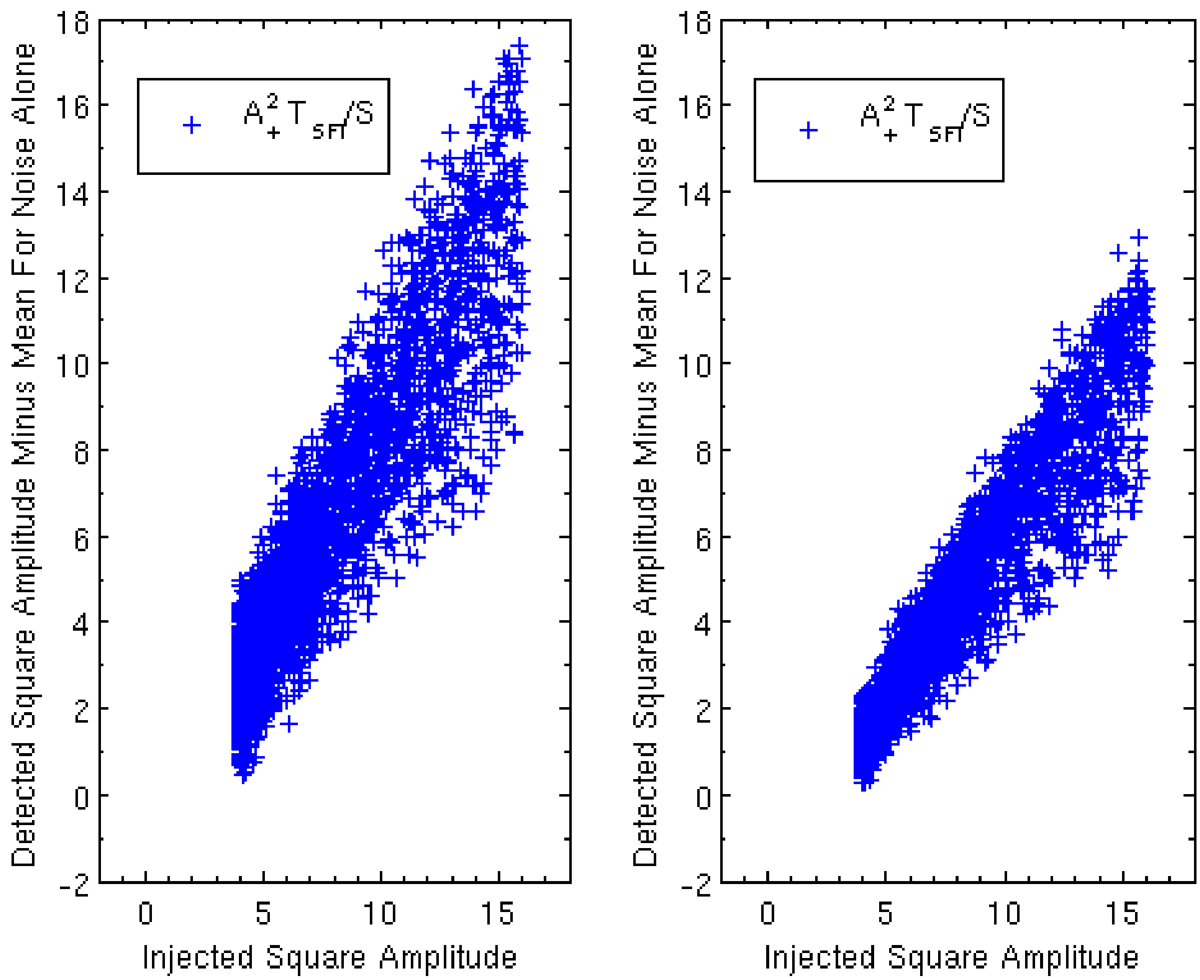

Figure 7: The figures show, for $h_0(T_{\rm SFT}/S)^{1/2} = 4$ and 3000 random signals, a comparison of the injected and detected amplitude A_+^2 for the PowerFlux Linear (polarization) method (left), and a comparison of the injected and detected amplitudes $A_+^2 = A_\times^2$ for the PowerFlux Circular (polarization) method (right).

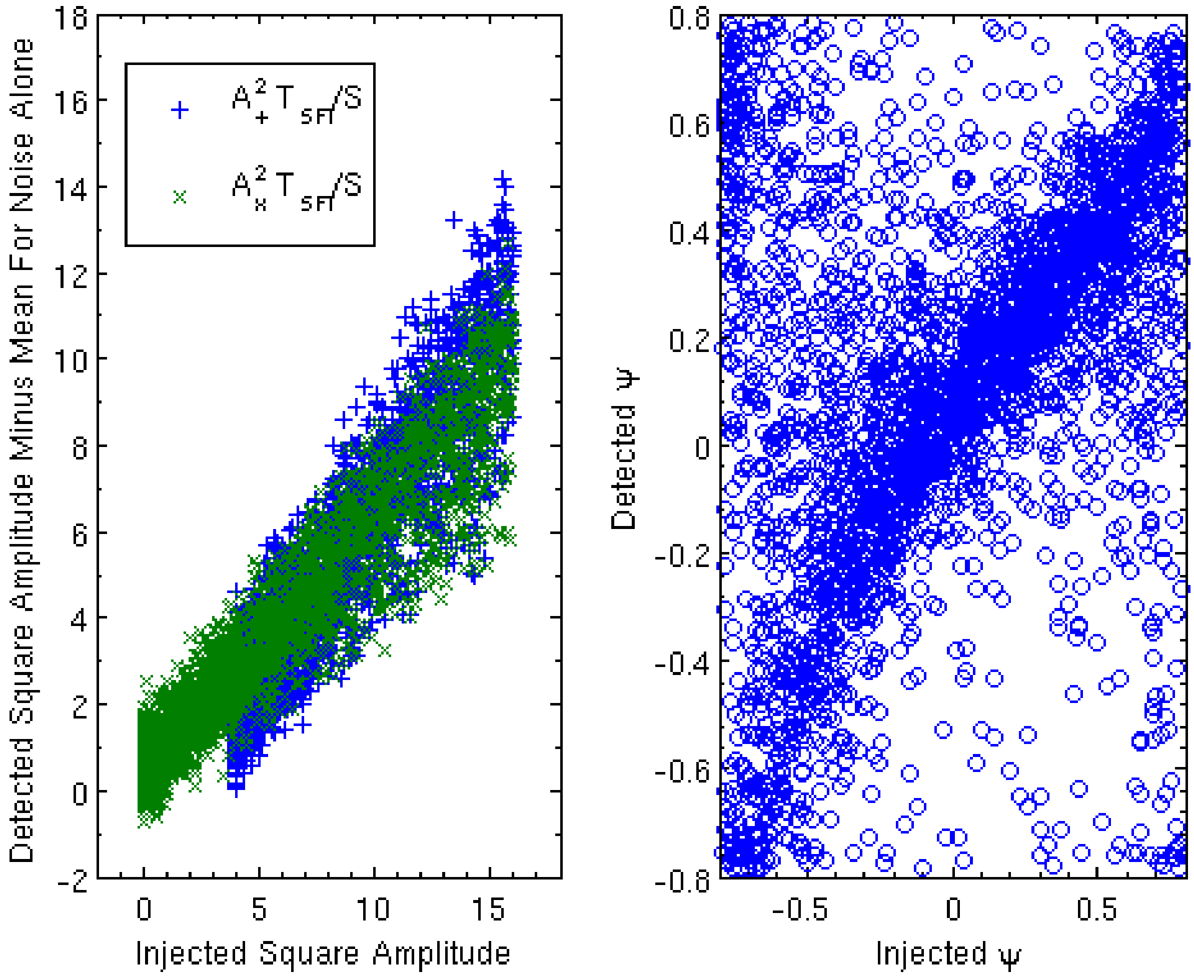

Figure 8: The figures show, for $h_0(T_{\mathrm{SFT}}/S)^{1/2} = 4$ and 3000 random signals, a comparison of the injected and detected amplitudes A_+^2 and $A_\times^2$ and the injected and detected ψ for PowerFlux Generalization II (left and right respectively).