Metropolis-Hastings Approach For Continuous Wave and Inspiral Searches

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Bayesian Inference, Posterior Computation and Parameter Estimation

Antenna output data $\mathbf{z}(\mathbf{t})$

Joint probability distribution function (PDF) $p(\mathbf{z}|\boldsymbol{\theta})$ conditional on unobserved parameters $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_d)$.

The PDF $p(\mathbf{z}|\boldsymbol{\theta})$ is usually referred to as the *likelihood* and regarded as a function of the parameters $\boldsymbol{\theta}$.

Bayes' Theorem: Condition on the post-experimental knowledge about $\boldsymbol{\theta}$ expressed through the *posterior* PDF

$$p(\boldsymbol{\theta}|\mathbf{z}) = \frac{p(\boldsymbol{\theta})p(\mathbf{z}|\boldsymbol{\theta})}{m(\mathbf{z})} \propto p(\boldsymbol{\theta})p(\mathbf{z}|\boldsymbol{\theta}) \tag{1}$$

$m(\mathbf{z}) = \int p(\mathbf{z}|\boldsymbol{\theta})p(\boldsymbol{\theta})d\boldsymbol{\theta}$ the marginal PDF of $\mathbf{z}$ and regarded as a normalizing constant.

Posterior PDF is proportional to product of prior and likelihood.

The point estimate of a single parameter, say θ_i, is the posterior mean

$$[\theta_i] = \int \theta_i p(\theta_i|\mathbf{z})d\theta_i \tag{2}$$

where

$$p(\theta_i|\mathbf{z}) = \int \dots \int p(\boldsymbol{\theta}|\mathbf{z})d\theta_1 \dots d\theta_{i-1}d\theta_{i+1} \dots d\theta_d. \tag{3}$$

is the marginal posterior PDF.

The Metropolis-Hastings Algorithm: Candidate is generated from an auxiliary PDF and accepted or rejected with some probability.

Candidate generating PDF, $q(\boldsymbol{\theta}|\boldsymbol{\theta}_n)$ depends on the current state $\boldsymbol{\theta}_n$ of the Markov chain. A new candidate $\boldsymbol{\theta}'$ is accepted with a certain *acceptance probability* $\alpha(\boldsymbol{\theta}'|\boldsymbol{\theta}_n)$ also depending on the current state $\boldsymbol{\theta}_n$ given by:

$$\alpha(\boldsymbol{\theta}'|\boldsymbol{\theta}_n) = \min\left\{\frac{p(\boldsymbol{\theta}')p(\mathbf{z}|\boldsymbol{\theta}')q(\boldsymbol{\theta}_n|\boldsymbol{\theta}')}{p(\boldsymbol{\theta}_n)p(\mathbf{z}|\boldsymbol{\theta}_n)q(\boldsymbol{\theta}'|\boldsymbol{\theta}_n)}, 1\right\}$$

if $(p(\boldsymbol{\theta}_n)p(\mathbf{z}|\boldsymbol{\theta}_n)q(\boldsymbol{\theta}'|\boldsymbol{\theta}_n)) > 0$ and $\alpha(\boldsymbol{\theta}'|\boldsymbol{\theta}_n) = 1$ otherwise. For good efficiency a multivariate normal distribution is used for $q(\boldsymbol{\theta}|\boldsymbol{\theta}_n)$. The steps of the MH algorithm are therefore:

Step 0: Start with an arbitrary value $\boldsymbol{\theta}_0$

Step $n+1$: Generate $\boldsymbol{\theta}'$ from $q(\boldsymbol{\theta}|\boldsymbol{\theta}_n)$ and u from $U(0,1)$

If $u \leq \alpha(\boldsymbol{\theta}'|\boldsymbol{\theta}_n)$ set $\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}'$ (acceptance)

If $u > \alpha(\boldsymbol{\theta}'|\boldsymbol{\theta}_n)$ set $\boldsymbol{\theta}_{n+1} = \boldsymbol{\theta}_n$ (rejection)

The signals were generated by Réjean Dupuis. Data consisted of 14000 points for $\mathbf{y}_k$, the heterodyned signal. Signals had the following parameters:

$\psi = 0.392699081$ $\phi = 0.785398163$ $\iota = 0.785398163$

RA = 0.0 DEC = 0.785398163

σ(y.re) = σ(y.im) = 1.0

The prior ranges used were:
$cos(\iota) -> [0, 1]$, $\psi -> [0, \pi/4], \phi -> [0, 2\pi], h -> [0, \text{inf})$
For results see
http://physics.carleton.edu/Faculty/Nelson/MCMCpulsar/MCMCpulsar.htm

Signals had the following parameters:

$\psi = 0.2$ $\phi = 0.1$ $\iota = 0.7$

RA = 0.0 DEC = 0.785398163

$\sigma(\text{y.re}) = \sigma(\text{y.im}) = 1.0$

The prior ranges used were:
$cos(\iota) -> [0, 1]$, $\psi -> [0, \pi/4], \phi -> [-\pi, \pi], h -> [0, \text{inf})$

What Next?

Primary aim is to conduct a $DirectedSearch$

Uncertainty (small at first)in the frequency f_0

After that - other parameters such as $\dot{f}_0$

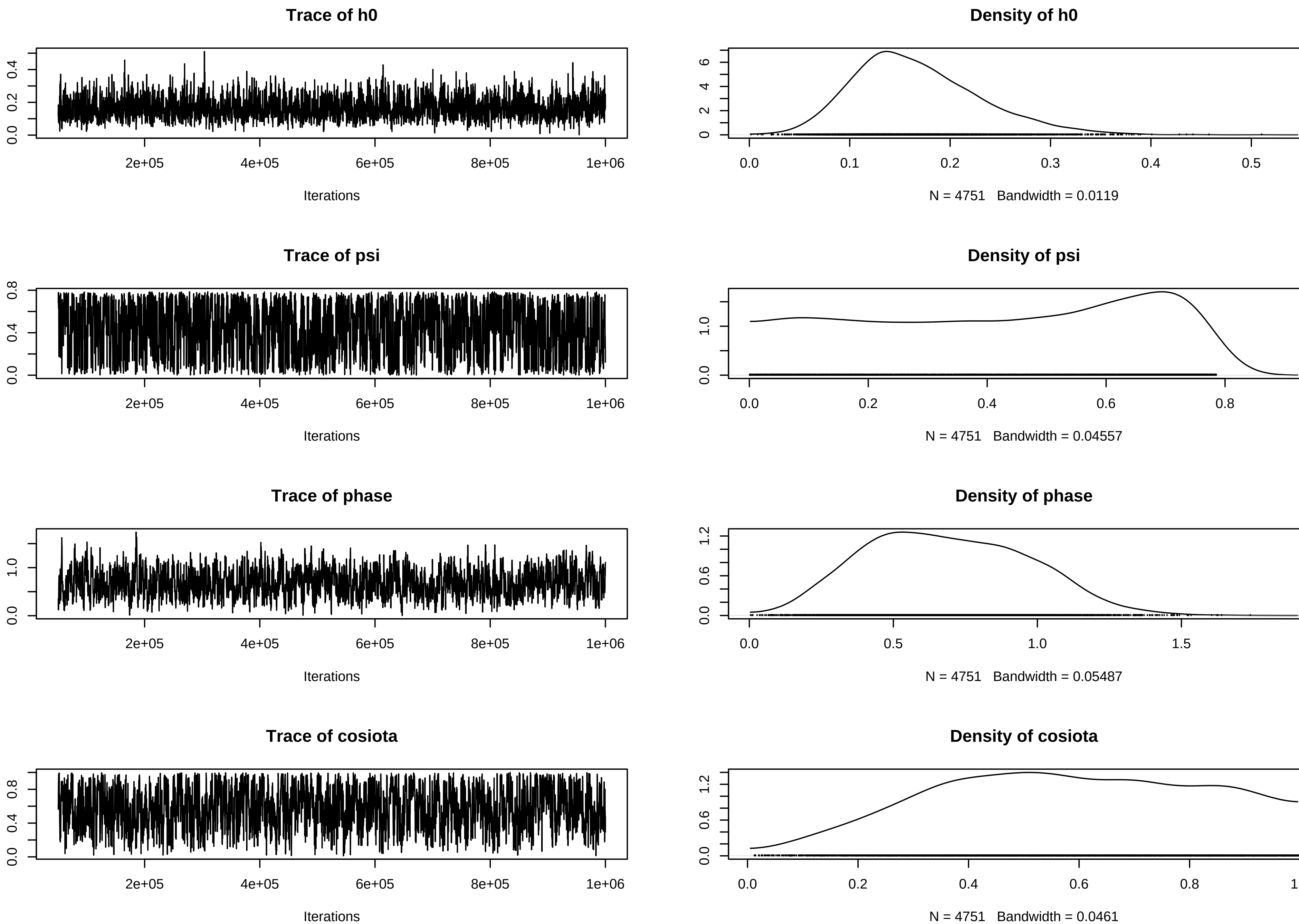
Sky Location???

Also Using Metropolis-Hastings

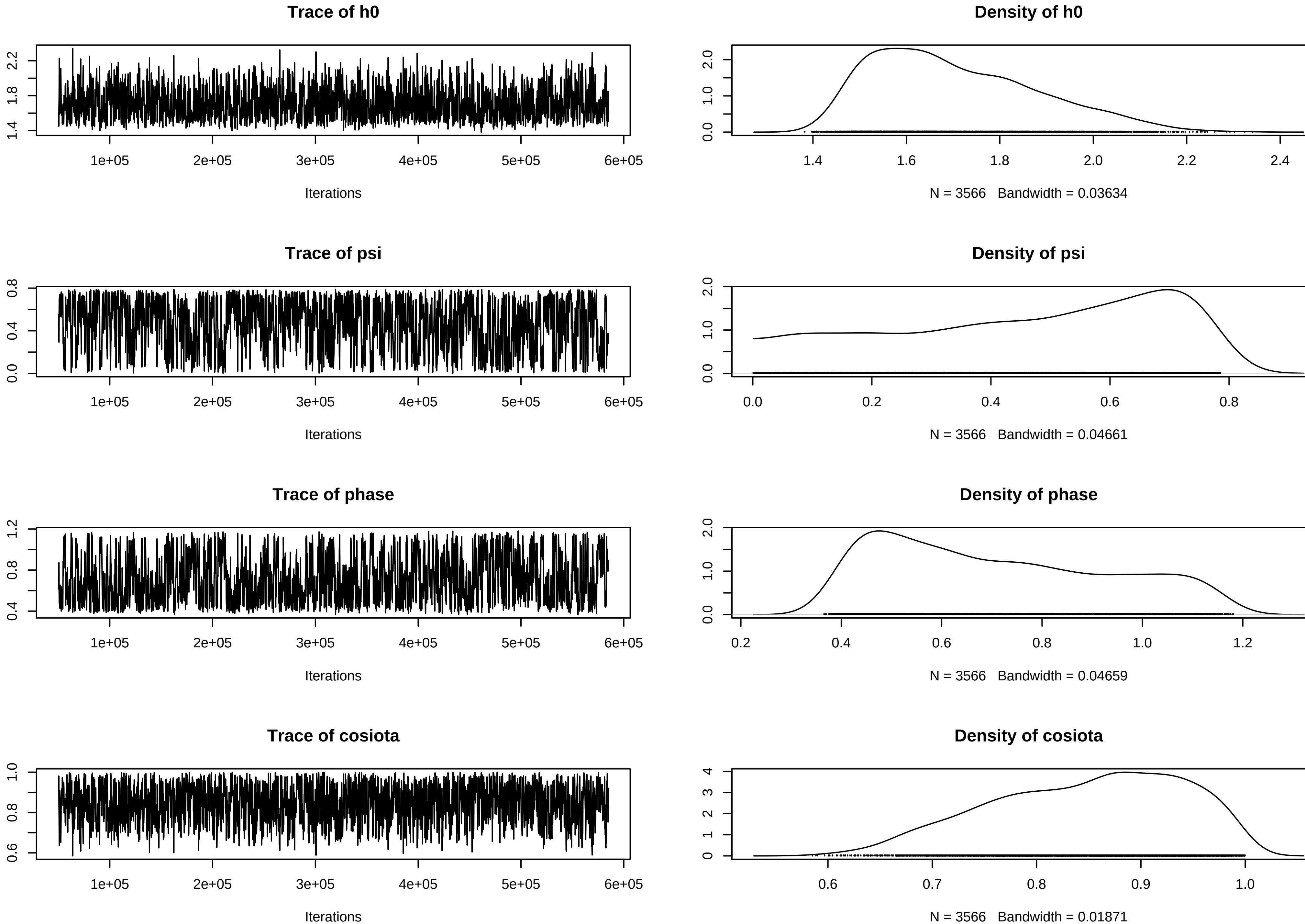
To Find Inspiral Events

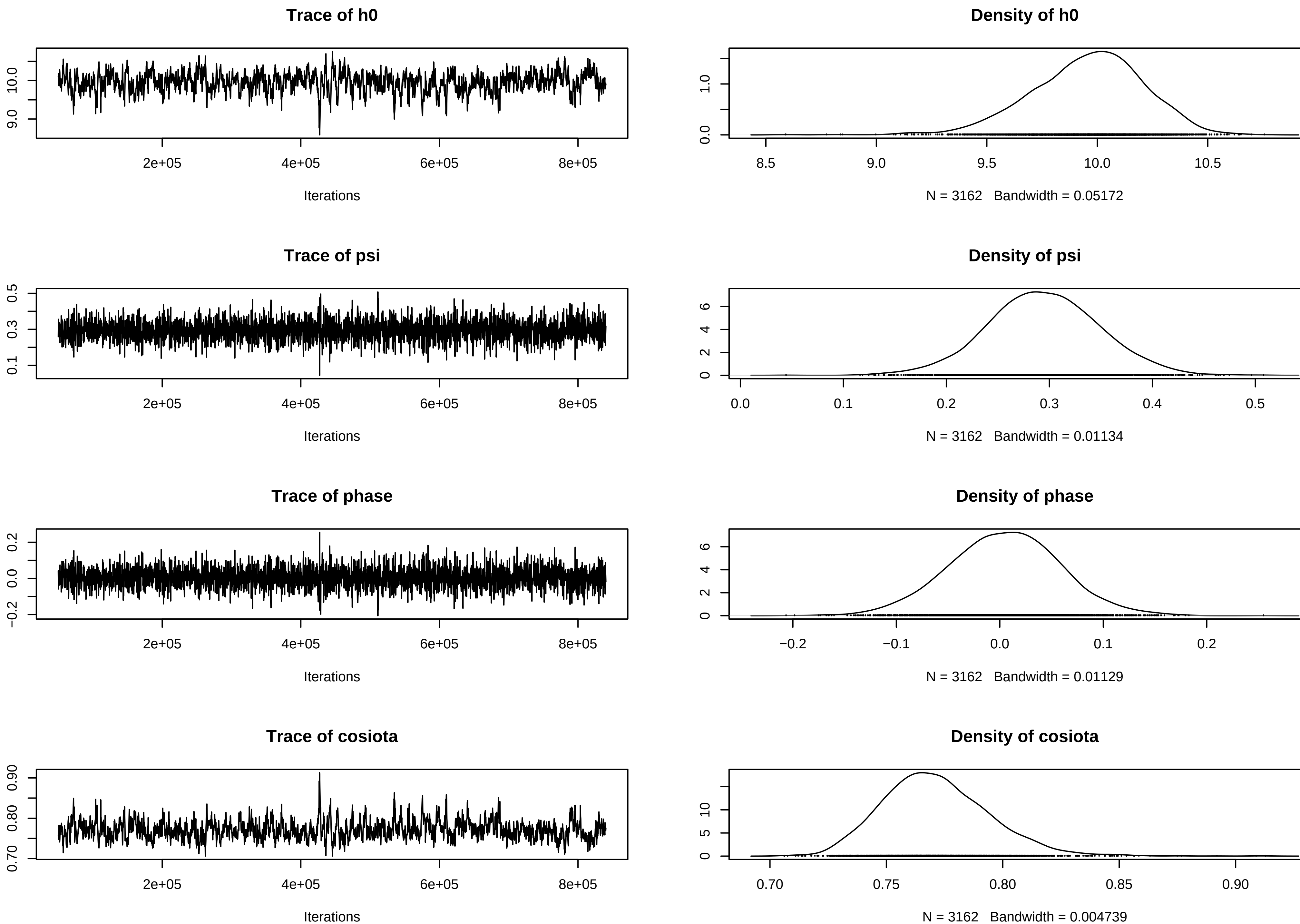
Example with $m_1 = 3.0$, $m_2 = 4.5$

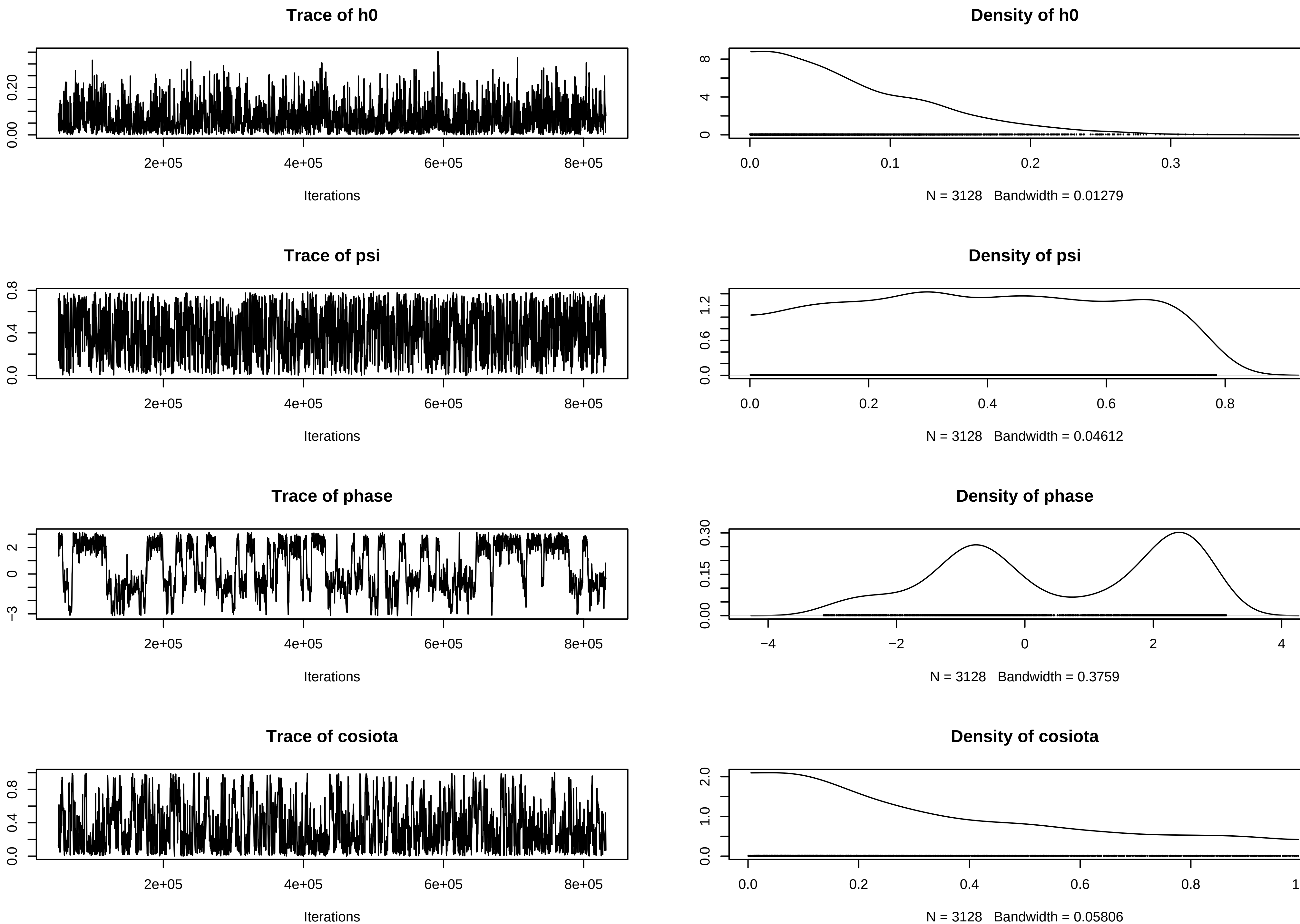
$\eta = 0.24$, $m_t = 7.5$, $SNR = 14$



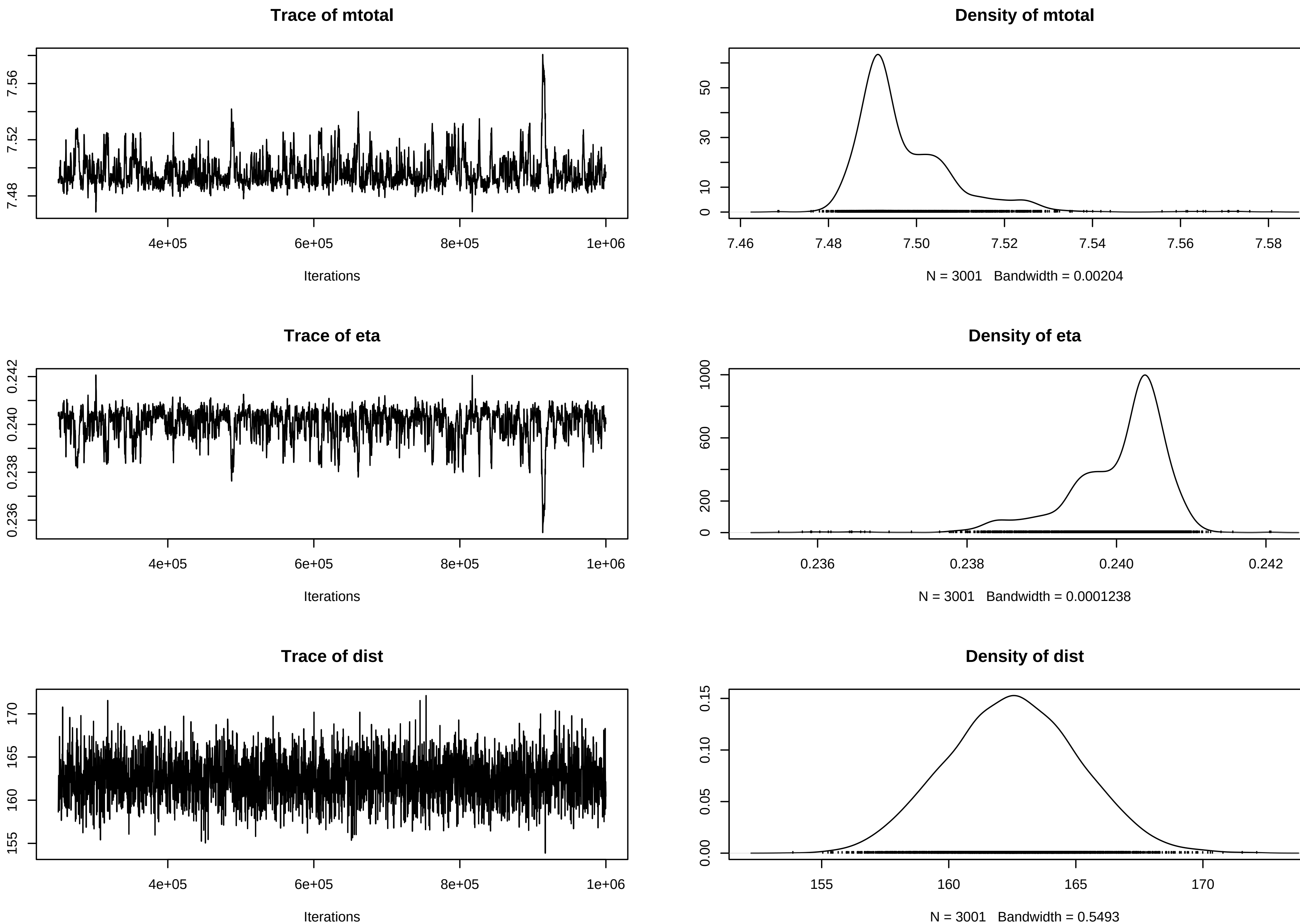
Trace of h0
Iterations
Density of h0
N = 4751 Bandwidth = 0.0119
Trace of psi
Iterations
Density of psi
N = 4751 Bandwidth = 0.04557
Trace of phase
Iterations
Density of phase
N = 4751 Bandwidth = 0.05487
Trace of cosiota
Iterations
Density of cosiota
N = 4751 Bandwidth = 0.0461


Trace of h0
Iterations
Density of h0
N = 3566 Bandwidth = 0.03634
Trace of psi
Iterations
Density of psi
N = 3566 Bandwidth = 0.04661
Trace of phase
Iterations
Density of phase
N = 3566 Bandwidth = 0.04659
Trace of cosiota
Iterations
Density of cosiota
N = 3566 Bandwidth = 0.01871


Trace of h0
Iterations
Density of h0
N = 3162   Bandwidth = 0.05172
Trace of psi
Iterations
Density of psi
N = 3162   Bandwidth = 0.01134
Trace of phase
Iterations
Density of phase
N = 3162   Bandwidth = 0.01129
Trace of cosiota
Iterations
Density of cosiota
N = 3162   Bandwidth = 0.004739



Trace of h0
0.00
0.20
2e+05
4e+05
6e+05
8e+05
Iterations
Density of h0
0
4
8
0.0
0.1
0.2
0.3
N = 3128   Bandwidth = 0.01279
Trace of psi
0.0
0.4
0.8
2e+05
4e+05
6e+05
8e+05
Iterations
Density of psi
0.0
0.6
1.2
0.0
0.2
0.4
0.6
0.8
N = 3128   Bandwidth = 0.04612
Trace of phase
−3
0
2
2e+05
4e+05
6e+05
8e+05
Iterations
Density of phase
0.00
0.15
0.30
−4
−2
0
2
4
N = 3128   Bandwidth = 0.3759
Trace of cosiota
0.0
0.4
0.8
2e+05
4e+05
6e+05
8e+05
Iterations
Density of cosiota
0.0
1.0
2.0
0.0
0.2
0.4
0.6
0.8
1
N = 3128   Bandwidth = 0.05806


Trace of mtotal
7.48
7.52
7.56
4e+05
6e+05
8e+05
1e+06
Iterations
Density of mtotal
0
10
30
50
7.46
7.48
7.50
7.52
7.54
7.56
7.58
N = 3001 Bandwidth = 0.00204
Trace of eta
0.236
0.238
0.240
0.242
4e+05
6e+05
8e+05
1e+06
Iterations
Density of eta
0
200
600
1000
0.236
0.238
0.240
0.242
N = 3001 Bandwidth = 0.0001238
Trace of dist
155
160
165
170
4e+05
6e+05
8e+05
1e+06
Iterations
Density of dist
0.00
0.05
0.10
0.15
155
160
165
170
N = 3001 Bandwidth = 0.5493