AN IMPLEMENTATION OF PINTO'S INTERPOLATION METHOD FOR CHIRPS.

ASIS

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— **The strategy**

- Let

$$h^X(t;\lambda_k), \quad k = 1, 2, ..., n_\lambda \tag{1}$$

be the exact waveform produced by nature (which we will call the *signal*), which is characterized by parameters λ_k.

- Let

$$h^A(t;\mu_k), \quad k = 1, 2, ..., n_\mu \tag{2}$$

be an approximate waveform (called a *template*) characterized by parameters μ_k.

- The *overlap* between $h^X(t;\lambda_k)$ and the time–shifted family $h^A(t-\tau;\mu k)$ is defined as

$$\mathcal{A}(\lambda_k, \mu_k) = \max_{\tau,\phi} \frac{< h^X(t;\lambda), h^A(t-\tau;\mu) >}{\sqrt{< h^X(t;\lambda), h^X(t;\lambda) >< h^A(t;\mu), h^A(t;\mu) >}} \tag{3}$$

Here the scalar product $< a, b >$ between two functions $a(t)$ and $b(t)$ is defined as

$$< a, b >= \int_{-\infty}^{\infty} \frac{\tilde{a}(f)\tilde{b}^*(f)}{S_n(f)} df \tag{4}$$

- The overlap between the signal and the template is maximized when the template is perfectly matched to the signal.

- There is a reduction in the overlap (and the SNR) obtained when the template does not necessarily match the signal.

- We need to space the templates throughout the parameter space so that whatever the parameters of the signal, it will produce an overlap with one of the templates which is greater than some given value (sometimes called the *minimal match*).

Newtonian Templates

- Let us take the simplest case where the approximate waveforms are the so called *Newtonian waveforms*.

- In this case, the parameter space describing the set of waveforms is one–dimensional, and may be characterized by the Newtonian *chirp time* τ_N,

$$\tau_N = \frac{5}{256}\mu^{-1}m^{-2/3}(\pi f_a)^{-8/3} \tag{5}$$

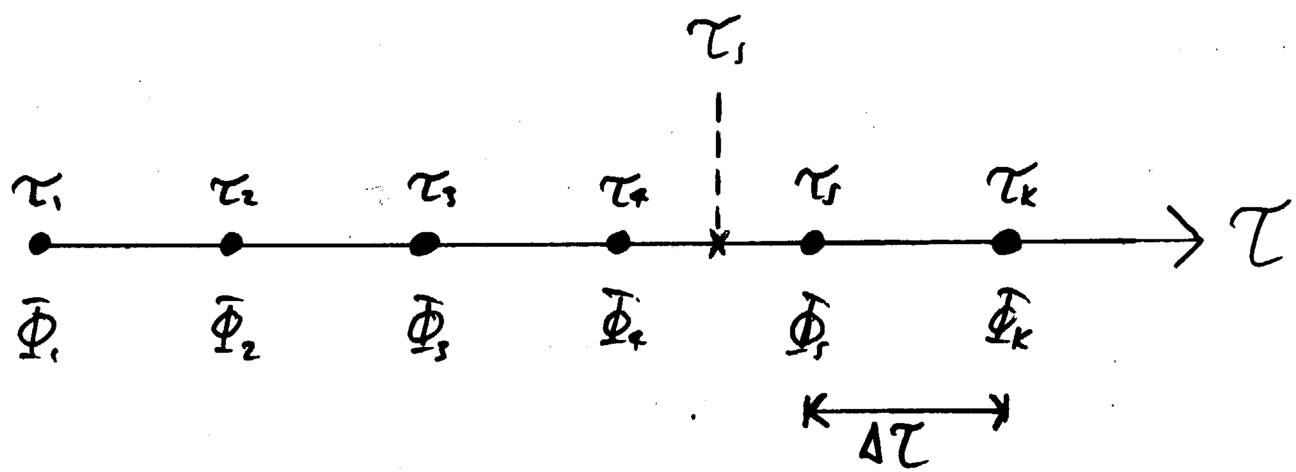
where τ_N represents the time taken for the frequency of the waveform to increase from $f = f_a$ to $f = \infty$. Here, $m = m_1 + m_2$ (the total mass of the binary), and $\mu = m_1 m_2/m$ (the reduced mass).

- Generate a *bank* of equally–spaced templates such that the chirp time of the k^{th} template is τ_k and the separation between any two neighbouring templates is $\tau_{k+1} - \tau_k = \Delta\tau$.

- Generate a signal which has chirp time τ_s which lies within the range covered by the templates.

- Calculate the overlap of the signal τ_s with each of the templates. Denote the overlap with the k^{th} template as Φ_k.

- The minimal match here is the overlap produced when a signal has a chirp time which lies mid–way between any two neighbouring templates.



The Sampling Theorum

- Let $h(t)$ be a function which is sampled at evenly spaced intervals in time with sampling interval Δ.

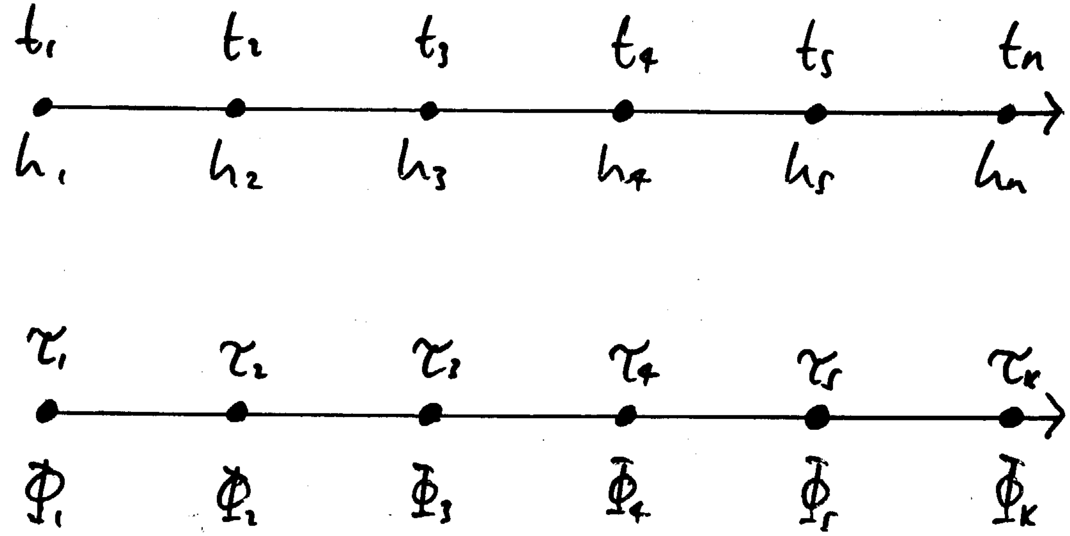
- The value of the function at a time $t_n = n\Delta$ is denoted h_n,

$$h_n = h(t = t_n) \ . \tag{6}$$

- The *sampling theorum* says that so long as the Fourier Transform of $h(t)$ falls to zero for $|f| > f_c$, where $f_c = 1/2\Delta$ then $h(t)$ can be re–constructed entirely from its discrete samples h_n, using

$$h(t) = \sum_{n=-\infty}^{\infty} h_n \ \mathrm{sinc}\left[\frac{\pi}{\Delta}(t - t_n)\right] \tag{7}$$

- A comparison:



$$h(t) = \sum_{n=-\infty}^{\infty} h_n \text{ sinc}\left[\frac{\pi}{\Delta}(t - t_n)\right] \tag{8}$$

$$\Phi(\tau) = \sum_{k=1}^{N_t} \Phi_k \text{ sinc}\left[\frac{\pi}{\Delta\tau}(\tau - \tau_k)\right] \tag{9}$$

A Numerical Experiment

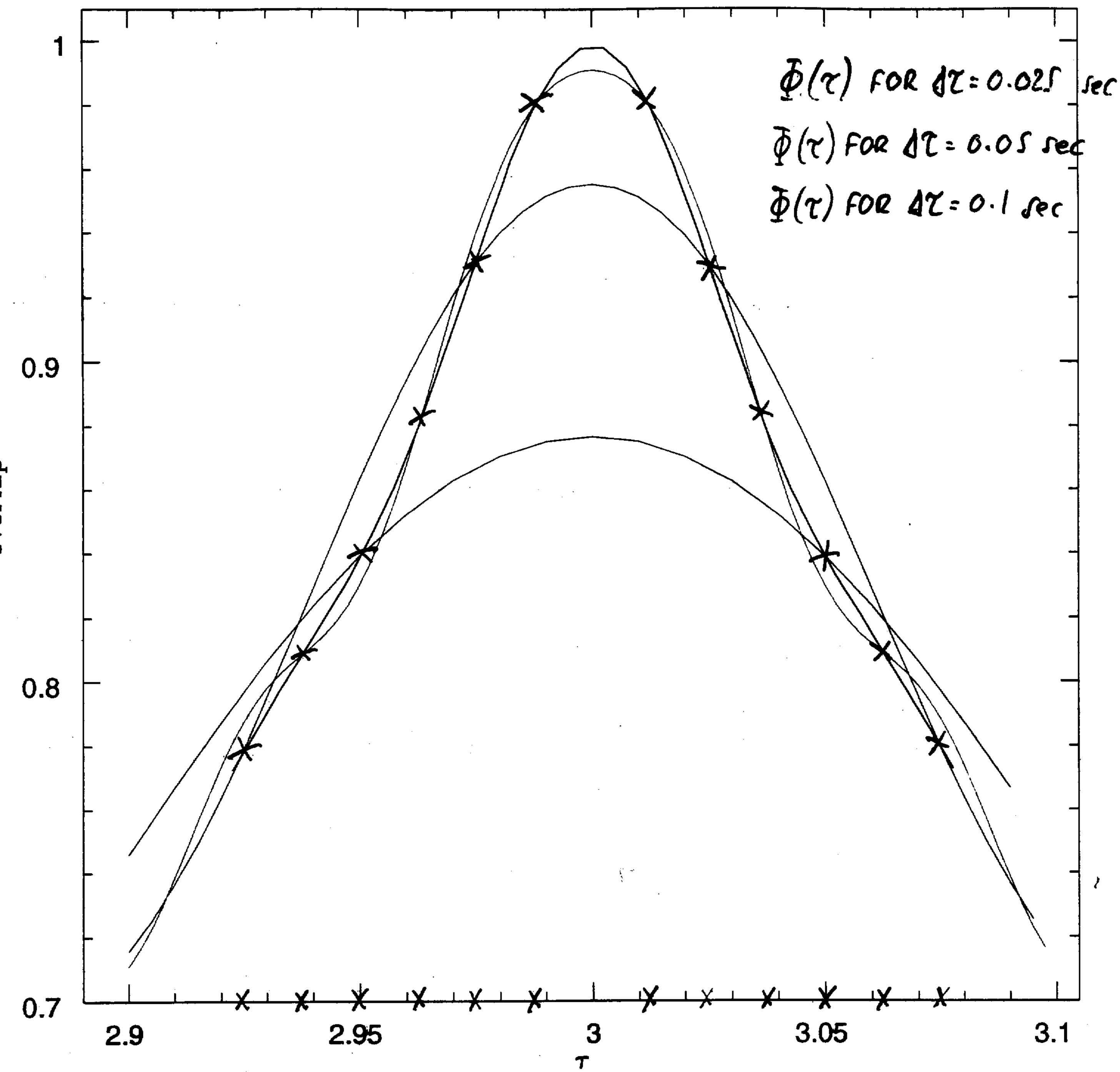
- Generate a signal waveform with a chirp time $\tau_s = 3.0$ seconds.

- Generate a bank of $N_t = 32$ templates, equally spaced either side of τ_s, with spacing $\Delta\tau$.

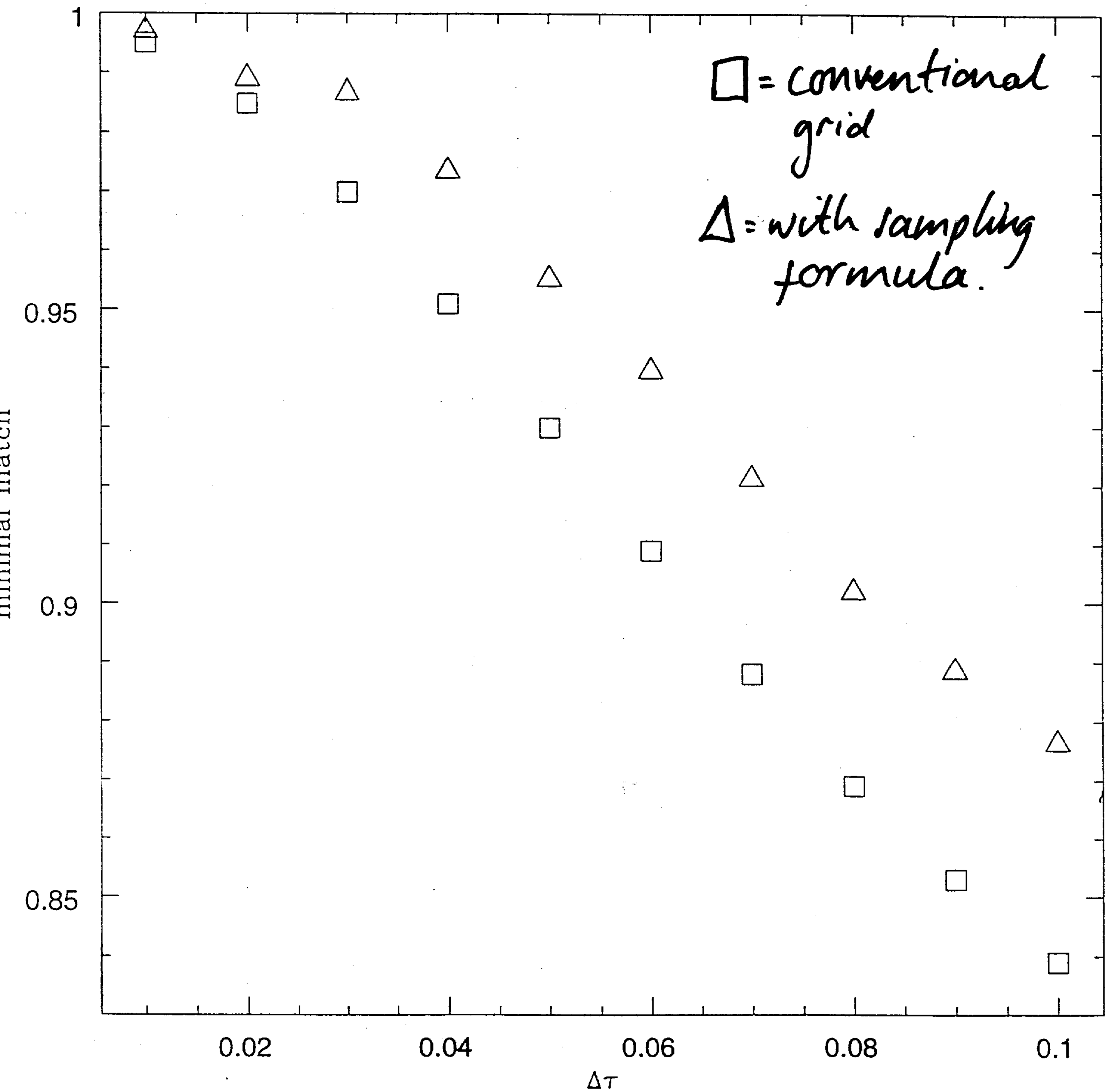
- Calculate the actual overlap between τ_s and each template τ_k to produce Φ_k.

- Calculate $\Phi(\tau)$, which is the expected overlap between the signal and any template which has a chirp time τ, using the sampling formula:

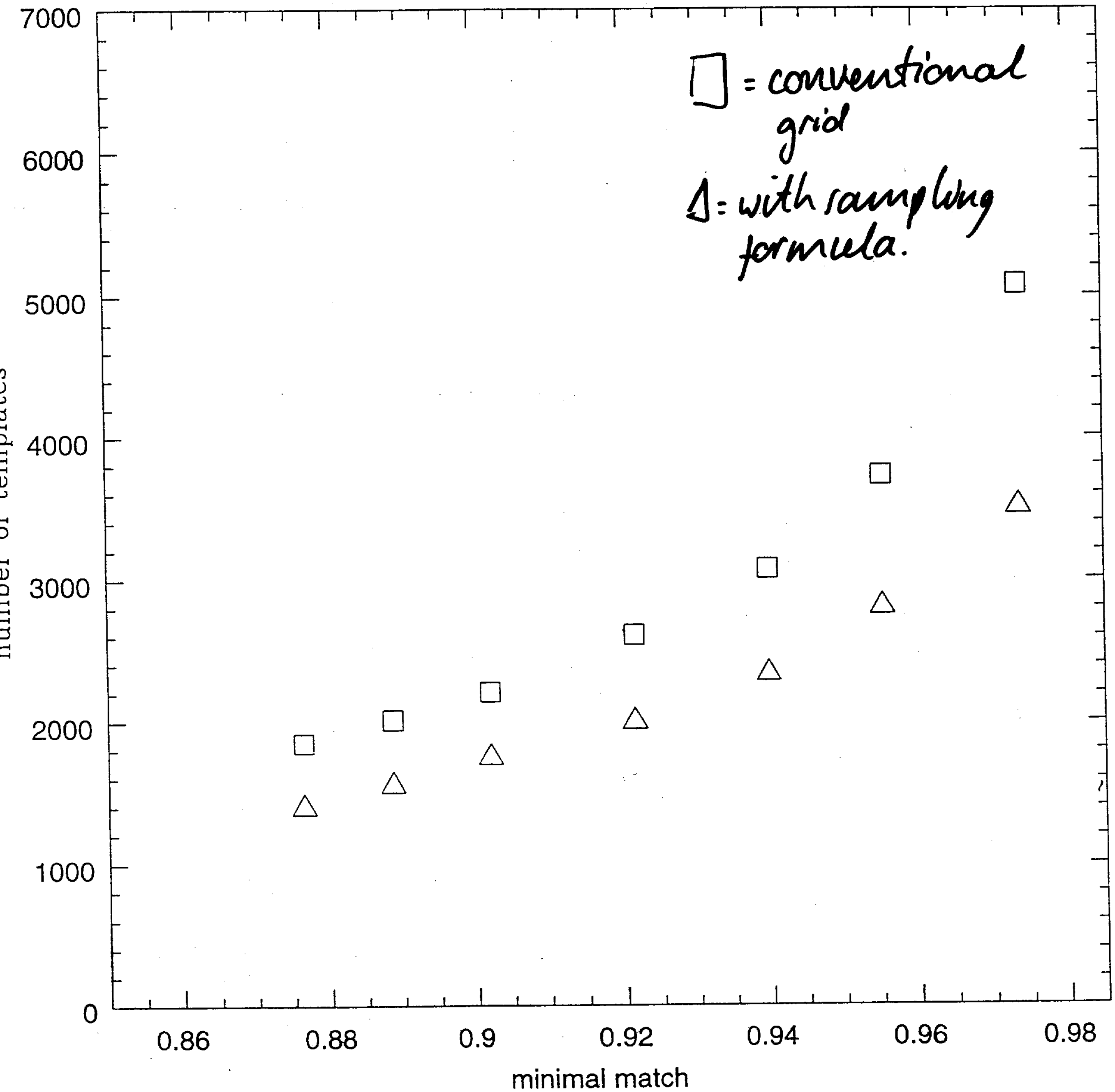
$$\Phi(\tau) = \sum_{k=1}^{N_t} \Phi_k \operatorname{sinc}\left[\frac{\pi}{\Delta\tau}(\tau - \tau_k)\right] \tag{9}$$



Φ(τ) FOR Δτ = 0.025 sec
Φ(τ) FOR Δτ = 0.05 sec
Φ(τ) FOR Δτ = 0.1 sec
1
0.9
0.8
0.7
2.9
2.95
3
3.05
3.1
τ



1
0.95
0.9
0.85
minimal match
0.02
0.04
0.06
0.08
0.1
Δτ
□ = conventional grid
Δ = with sampling formula.



□ = conventional grid
Δ = with sampling formula.
7000
6000
5000
4000
3000
2000
1000
0
0.86
0.88
0.9
0.92
0.94
0.96
0.98
minimal match

- The magnitude of the peak in these curves gives us the overlap produced by the template bank when the signal lies mid–way between any two of the templates.
  - This was our earlier definition of the minimal match.

- Therefore, for a given template spacing $\Delta\tau$, we can:
  - Calculate the minimal match one would get just using a conventional bank of templates
  - Calculate the minimal match one can get with the application of the sampling formula to the conventional bank

- Equivalently, for a given minimal match, we can calculate the template spacing $\Delta\tau$ which would be needed to produce that minimal match.
  - For the conventional grid
  - For the conventional grid with the application of the sampling formula

- From this, we can calculate the number of templates needed to search the whole parameter space in each case.

Conclusions

- For a given template spacing $\Delta\tau$, the minimal match obtained is increased upon use of the sampling formula

  — e.g. for $\Delta\tau = 0.04$ sec,

  $$(\text{mimimal match})_{conv} = 0.951$$

  $$(\text{mimimal match})_{sampl} = 0.973$$

- For a given minimal match, the number of templates needed to cover the parameter space is reduced upon use of the sampling formula

  — e.g. for a minimal match of 0.97,

  $$N_{T\,conv} = 5070$$

  $$N_{T\,sampl} = 3500$$

  — which is a reduction by a factor of ~ 1.45.

Note 1, Linda Turner, 05/09/00 09:06:53 AM
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