LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
- LIGO -
CALIFORNIA INSTITUTE OF TECHNOLOGY
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Technical Note **LIGO-T970058-00 - D** 2/11/97

Modal Model Update 7

Angular Transfer Functions

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Distribution of this draft:

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1 Abstract

For LIGO the angular transfer functions as measured by the wavefront sensors are ‘flat’ up to frequencies where the TEM_{10} mode becomes resonant in one of the arm cavities.

2 Audio Sidebands

One way to overcome the limitations of the static modal model is to extend the modal space by including audio sideband frequencies. This technique allows to accurately simulate stationary configurations without the numerical costs of an approach involving differential (or difference) equations. It does not allow to study transients or the propagation of non-stationary noise, but it is well suited to determine (open loop) transfer functions. The only change affects the basic operators, such as the free space propagator and the mirror distortion operators, whereas cavity equations and resonance condition stay exactly the same, however, acting on a larger space now.

2.1 Modal Space

We extend the modal space by replacing every element of a modal state vector with an array of audio frequencies, i.e. we form a vector where each element has two indices describing the mode order and the frequency order.

$$\vec{E} = \begin{bmatrix} TEM_{00} \\ TEM_{10} \\ TEM_{01} \\ \dots \end{bmatrix} \quad \text{with} \quad TEM_{mn} = \begin{bmatrix} E_0 \\ E_f \\ \dots \\ E_{-f} \end{bmatrix} \tag{1}$$

where the frequencies are ordered in a ‘two’s-complements’ way; where the next element in the vector is always the next higher harmonics, except for the highest harmonics which is followed by the lowest one (most negative). Note that we use negative frequencies to account for cosine and sine terms.

2.2 Propagator

Since mode order and frequency order commute for free-space propagation, the propagator can be written as a block matrix, where each block describes the phase shift induced by the audio frequency and where each block is identical for every mode order. Up to a factor of e^{-ikl} one can write the propagator as

$$P = \begin{bmatrix} P_f e^{i\eta} & 0 & \dots \\ 0 & P_f e^{2i\eta} & 0 \\ \dots & 0 & \dots \end{bmatrix} \quad \text{with} \quad P_f = e^{-iP_T} \quad \text{and} \quad P_T = \begin{cases} r\,2\pi f \dfrac{l}{c} & \text{if } r = s \\ \dfrac{k\Delta l}{2} & \text{if } |r-s| = 1 \\ 0 & \text{else} \end{cases} \tag{2}$$

where η is the guoy phase shift and where r and s denote the frequency order and not the row or column number. k is the wave vector of the light, f the fundamental frequency of the audio sidebands, l the propagation length, c the speed of light and Δl the amplitude of the length wiggle at frequency f (cosine term). From the form of eqn. (2) one can see the on-diagonal terms account for the additional phase shift an electric field at an audio sideband frequency experiences when propagating over the length l. The off-diagonal terms represent the amount of field transferred into the next higher and the next lower audio sideband when the length is dithered with a cosine of amplitude Δl and frequency f. If only one frequency order is needed, the matrices will be of order 3, i.e.

$$P_T = \begin{bmatrix} 0 & \dfrac{k\Delta l}{2} & \dfrac{k\Delta l}{2} \\ \dfrac{k\Delta l}{2} & 2\pi f \dfrac{l}{c} & 0 \\ \dfrac{k\Delta l}{2} & 0 & -2\pi f \dfrac{l}{c} \end{bmatrix} \tag{3}$$

2.3 Mirror Distortion and Angular Misalignment

Following ref. [1] one can write a mirror tilt operator as the exponential function of a Hermitian operator:

$$M = e^{-i\sqrt{8}\Theta T} \tag{4}$$

where Θ is the normalized misalignment angle and T is the generator associated with a misalignment. Since, again, mode order and frequency order commute, one can write the misalignment generator as a block matrix of the form:

$$T = T_{mn,\,kl} T_f \tag{5}$$

where $T_{mn,\,kl}$ stands for the 'old' modal generator and T_f for the audio frequency matrix. If one writes the misalignment angles as

$$\Theta(t) = \Theta_S + \Theta_D \cos 2\pi f t \tag{6}$$

where Θ_S is a static tilt and Θ_D is the amplitude of an angular dither at frequency f. The audio frequency matrix becomes:

$$T_f = \begin{cases} \Theta_S & \text{if } r = s \\ \dfrac{\Theta_D}{2} & \text{if } |r-s| = 1 \\ 0 & \text{else} \end{cases} \tag{7}$$

where r and s, again, denote the frequency order and not the row or column number. If only one frequency order is needed, the audio frequency matrices becomes

$$P_T = \begin{bmatrix} \Theta_S & \dfrac{\Theta_D}{2} & \dfrac{\Theta_D}{2} \\ \dfrac{\Theta_D}{2} & \Theta_S & 0 \\ \dfrac{\Theta_D}{2} & 0 & \Theta_S \end{bmatrix} \tag{8}$$

2.4 DEMODULATOR

The demodulator operator is an operator which describes the shape of the photodetector through its modal projection and, when audio sidebands are added, extracts the desired audio frequencies. One can then write

$$D^{\Omega} = D^{\Omega}_{mn,\,kl} D_t \tag{9}$$

where Ω stands for the shape of the photodetector, $D^{\Omega}_{mn,\,kl}$ is the 'old' modal space demodulator operator and D_t is the audio frequency demodulation operator which down-converts the modulation frequency f_m offseted by an audio frequency of order t, i.e $t \times f$.

$$D_t = \begin{cases} 1 & \text{if } s - r = t \\ 0 & \text{else} \end{cases} \tag{10}$$

with r and s denoting the frequency order. If only one frequency order is needed, the three demodulators down-converting the frequencies at $f_m - f$, f_m and $f_m + f$ become

$$D_{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad D_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{11}$$

3 Angular Transfer Functions

Angular misalignments generates TEM_{10} mode which propagates through the interferometer and is eventually detected by a wavefront sensor. During normal operation both arm cavities and the recycling cavity are resonant for the TEM_{00} mode, but only the recycling cavity is also resonant for the TEM_{10} mode due to its degeneracy. This means that one would not expect to see any frequency dependence of the angular transfer functions up to the frequency where the TEM_{10} mode gets resonant in the arm cavity. This happens at 11397 Hz (guoy frequency in the arm cavity) and repeats each 37483 Hz (free-spectral-range of the arm cavity, FSR). Because audio sidebands are on both sides of the carrier, it also happens at –11397 Hz which then wraps around to 26087 Hz.

Fig. 1 shows the angular transfer functions of an ETM dither as measured by the wavefront sensor at the dark port and as measured by one of the wavefront sensors in reflection using the non-resonant sideband at a 90° guoy phase shift. One main difference between the two transfer functions is the width of the resonances: at the dark port the width is determined by the finesse of the arm cavities alone, whereas in reflection the width is determined by the width of the double resonance. An other feature is the notch at half the FSR (and repeated every FSR). Strictly speaking, the signal does not disappear at this frequency, but rather shifts into the orthogonal guoy phase. Conversely, a signal which is in the orthogonal guoy phase at dc becomes measurable at higher frequencies. In LIGO this effect can be best seen in reflection using the non-resonant sideband, where the common ETM and ITM signals are at 90° guoy phase shift and the RM signal is at 0° guoy phase shift (see Fig. 2). However, below 1 kHz this effect is never dominant and completely irrelevant for the angular servo loops which have a bandwidth of only a couple of Hz.

Looking at all wavefront sensor signals and all angular degrees-of-freedom at each port we conclude that there is no frequency dependence of the wavefront sensor signals which would be significant for the angular alignment servos.

Reference

[1] Y. Hefetz, N. Mavalvala and D. Sigg, *“Principles of calculating alignment signals in complex optical interferometers“*, LIGO-P960024-A-D (1996).

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[9] D. Sigg, *“Modal model update 6: Mode Cleaner”*, LIGO-T960118-00-D (1996).

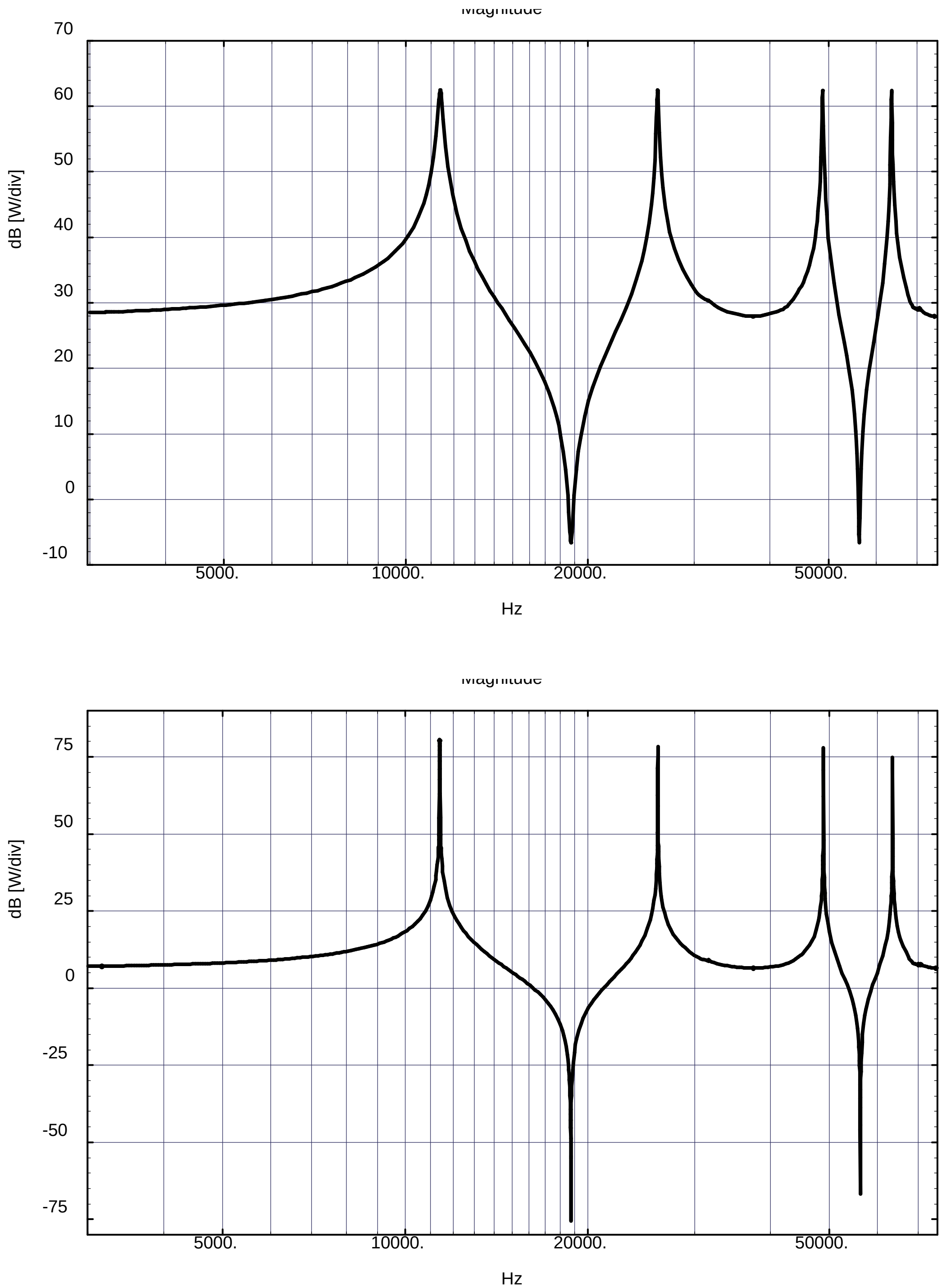

Figure 1: Angular transfer function of a common ETM dither measured at the dark port (top) and in reflection. The resonances are determined by the Guoy frequency of 11397 Hz and the FSR of 37483 Hz of the arm cavities. The resonance in reflection are narrower because of the double resonance.

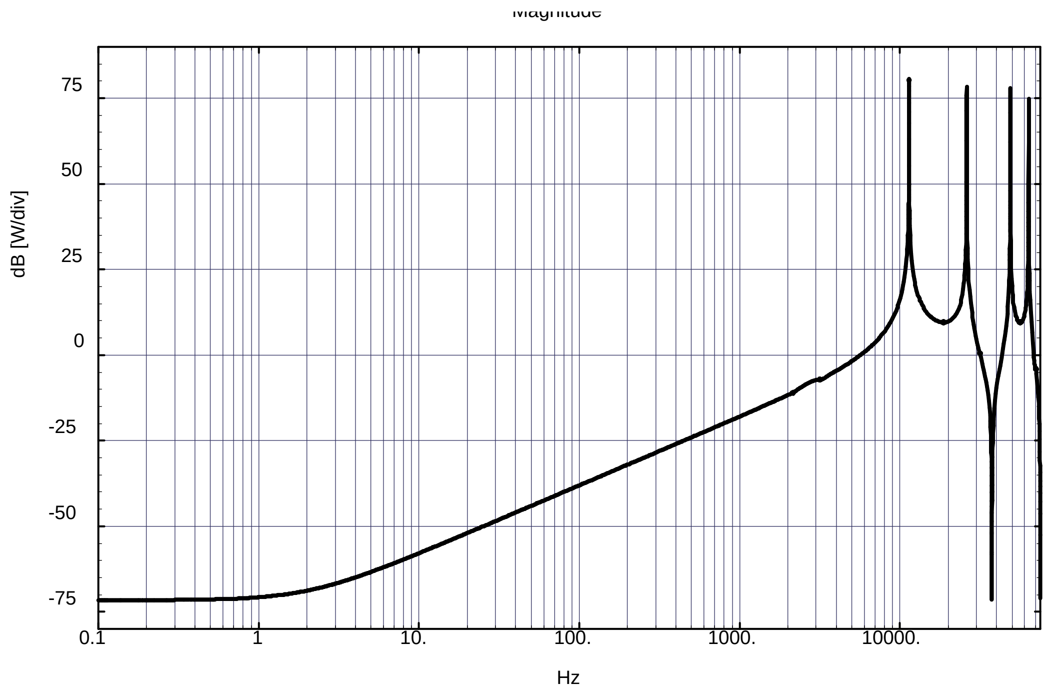

Figure 2: Common ETM dither measured in reflection using the non-resonant sideband as it is appearing in the orthogonal guoy phase.