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Modal Model Update 3

Small Angle Regime

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1 ABSTRACT

This document investigates the effects of small misalignments in a LIGO interferometer configuration. Section 3 looks at the wavefront sensor signals from angular misalignments of the interferometer mirrors and from tilts and shifts of the input laser beam. As a result a robust alignment matrix is presented [3], using the angles of the test masses and the recycling mirror only. The beamsplitter angle and the input beam deviations are then written as linear combinations of the other degree-of freedoms.

In section 4 the light levels circulating in the interferometer are calculated, which are then used in section 5 to deduce the shot noise limited detection angles. As it turns out, the shot noise limited detection angles are tiny and of no practical importance.

Section 6 investigates the effects of an angular misalignment on the beam centering. A centering matrix is deduced which can be used to control the beamsplitter angle and the input beam direction and offset from the centering information obtained in transmission of the end test masses and in reflection from the recycling mirror.

In section 7 the second order effect of a direct coupling of an angular misalignment into a gravitational wave signal is calculated; and in section 8 the same is done for beam jitter. The angular jitter at 150 Hz must not be larger than $2\times10^{-16}\ \text{rad}/\sqrt{\text{Hz}}$, otherwise the gravitational wave signal will be deteriorated. The input beam jitter at 150 Hz has to be smaller than $3\times10^{-14}\ \text{rad}/\sqrt{\text{Hz}}$ for a tilt and smaller than $7\times10^{-10}\ \text{m}/\sqrt{\text{Hz}}$ for a lateral shift to keep its effect on the gravitational wave signal small enough.

2 DEFINITIONS AND PARAMETERS

The basic formalism of the modal model as used in this document is outlined in ref. [1], whereas the extensions and the conventions used for the LIGO interferometer configuration are given in ref. [2]. The interferometer parameters used for the calculations in this paper are listed in Table 1.

Table 1: Interferometer parameters. LIGO configuration.

Parameter	Unit	arm (ITM)	arm (ETM)	recycl. (RM)
length (common / differential)	m	4000		7.533 / 0.14
power transmission	%	3	0.02	4
losses	ppm	0	200	0
radius of curvature	m	–14540	7400	–9851
modulation frequencies	MHz	39.848		19.918
modulation depths	G	0.5		0.05
wave length	µm	1.06		

3 ALIGNMENT MATRIX

The alignment matrix for the LIGO interferometer configuration was first calculated in ref. [3]. Following [4], we write the AM modulated light signal measured with a half-plane photodetector (see Appendix A.3 of [1]) at an extraction port as follows:

$$WFS(t, \eta, \Theta) = 2J_0(\Gamma)J_1(\Gamma)Pf_{\text{split}}\,k_{PD}^{10}\sum_i A_i\Theta_i\cos(\eta-\eta_{0i})\cos(\omega_m t+\phi_{0i}) \tag{1}$$

where P is the input laser power, J_0 and J_1 are the Bessel functions which describe the amplitude of the carrier and its sidebands for a given modulation depth Γ, f_{split} is the amount of power which is split off for the wavefront sensor, k_{PD}^{10} is a factor which describes the exact shape of the photodetector and is unity for a half-plane detector, A_i are the signal amplitudes, Θ_i are the misalignment angles of the mirrors, η is the Guoy phase of the fundamental mode between the extraction port and the photodetector, η_{0i} are the intrinsic signal Guoy phase shifts, ω_m is the modulation frequency and ϕ_{0i} are the intrinsic signal rf phase shifts. From eqn. (1) one can easily obtain the photocurrent induced in the detector by multiplying with the photodiode efficiency.

Table 2: Wavefront sensor signals. Top entry in each cell is A_i (with significant values in boldface), lower-left is rf-phase, and lower-right is the guoy phase η_{0i}.

	angular degree-of-freedom									
port	**ΔETM**		**ΔITM**		**ETM**		**ITM**		**RM**	
Dark	**–21.4**		**–9.77**		< 10^{-3}		< 10^{-3}		< 10^{-3}	
	Q	90°	Q	90°	Q	0°	Q	0°	Q	90°
Reflected	3.9×10^{-2}		**–2.31**		–0.252		**13.1**		**–18.4**	
	Q	145°	Q	145°	I	45°	I	144°	I	145°
Recycling cavity / 1000	5.3×10^{-3}		**–0.318**		5.0×10^{-2}		**1.82**		**–2.53**	
	Q	145°	Q	145°	I	61°	I	144°	I	145°
On-line arm reflection / 1000	-6.2×10^{-2}		**0.164**		2.5×10^{-2}		**0.922**		**–1.28**	
	Q+4°	~90°	I–12°	143°	Q+5°	61°	Q+5°	144°	Q+5°	145°
Reflected (non-resonant sideband)	< 10^{-3}		< 10^{-3}		**–2.19**		**–0.997**		**2.00**	
	I+5°	0°	Q+5°	0°	I	90°	I	90°	I	0°

The wavefront sensor signals measured at different extraction ports give different combinations of misalignment angle signals, and form the so-called alignment matrix. If this matrix is non-singular, i.e. the it is always possible to determine which mirror caused the misalignment, one has a robust sensing scheme for an angular servo system.

Table 2 lists the wavefront sensing signals at the dark port, in reflection, at the recycling cavity pick-off, in reflection of on-line arm and in reflection for a non-resonant sideband. Listed are the amplitudes, the intrinsic Guoy phase shifts and the intrinsic rf phase shifts. The letter ‘I’ denotes the ‘in’-phase (0°), whereas ‘Q’ denotes the ‘quad’-phase (90°).

A possible alignment matrix is presented in Table 3, using wavefront sensors in reflection and at the dark port only. This matrix is non-singular and, thus, provides a robust sensing scheme for the angular misalignment servo system.

Table 3: Possible setup of wavefront sensors. The first three columns describe the position, the rf phase and the Guoy phase of each detector. To the right the corresponding signal amplitudes are listed for an independent set of angular degree of freedoms, forming the alignment matrix.

	phases		**angular degrees of freedom**				
port	**rf**	**Guoy**	**ΔETM**	**ΔITM**	**ETM**	**ITM**	**RM**
1 dark port	Q	90°	**–21.4**	**–9.77**	0	0	0
2 reflection	Q	145°	0.039	**–2.31**	0	0	0
3 reflection	I	145°	0	0	–0.044	**13.1**	**–18.4**
4 reflection, NR	I	90°	0	0	**–2.19**	**–0.977**	0
5 reflection, NR	I	0°	0	0	0	0	**2.00**

Table 4: Wavefront sensing signals for a beamsplitter misalignment (BS) and for a tilt or shift of the input laser beam. Top entry in each cell is A_i (with significant values in boldface), lower-left is rf-phase, and lower-right is the guoy phase η_{0i}.

	BS		**input beam**			
port	**angle**		**tilt**		**shift**	
Dark	–0.147		$< 10^{-3}$		$< 10^{-3}$	
	Q	28°	Q	90°	Q	31°
Reflected	**–18.9**		**0.241**		**–0.259**	
	I–10°	145°	I	127°	I	67°
Recycling cavity / 1000	**–2.61**		4.6×10^{-2}		-5.0×10^{-2}	
	I–10°	145°	I	145°	I	85°
Arm reflection / 1000	**–1.28**		2.3×10^{-2}		-2.5×10^{-2}	
	Q–5°	143°	Q+5°	145°	Q+5°	85°
Reflected – NR sideband	–0.022		**2.02**		**2.17**	
	I–2°	19°	I	0°	I	120°

One interesting feature of this matrix is that the two wavefront sensors 1 and 3 measure approximately the same linear combination of misalignment angles which are most sensitive to reduce the signal-to-noise ratio of the gravitational-wave readout [5]. One can also see that the non-resonant sidebands are absolutely essential to distinguish a misalignment of the recycling mirror from a common ITM misalignment. It also a better way to detect the common ETM misalignment which is usually a rather small effect when measured with other wavefront sensors. It is instructive to see how the non-resonant sideband actually does its trick. Normally, a highly degenerate single cavity will produce similar alignment signals for the input and the rear mirror. So, one might expect that it is impossible to ever fully decompose the misalignment signals in the LIGO recycling cavity. But, a coupled double cavity has the unique property that a tilt of the middle mirror changes the resonant eigenmode in the rear cavity and that, thus, the reflected light from the middle mirror (rear cavity) is also laterally shifted with respect to the incident beam. It is exactly this shift vs. tilt which then separates the two signals from the front and the middle mirror in Guoy phase.

Until now, only angular misalignments of the test masses and the recycling mirror were investigated, and the influence of a misaligned beamsplitter or tilted and shifted input laser beam was neglected. These are not new angular degree-of-freedoms and both the misalignment of the beamsplitter and the input beam can be expressed as linear combinations of a misalignment of the four test masses and the recycling mirror. The wavefront sensor signals are listed in Table 4. As one would expect the beamsplitter misalignment (BS) is a combination of the off-line arm mirror angles only:

$$\mathrm{BS} = 0.92\ \mathrm{ETM}_2 - 2.01\ \mathrm{ITM}_2 - 0.01\ \mathrm{RM} \tag{2}$$

Furthermore, a tilt of the input beam $\mathrm{IB}_{\mathrm{tilt}}$ can be written as a common misalignment of all interferometer mirrors

$$\mathrm{IB}_{\mathrm{tilt}} = -0.652\ \overline{\mathrm{ETM}} + 1.431\ \overline{\mathrm{ITM}} + 1.01\ \mathrm{RM} \tag{3}$$

whereas a shift of the input beam $\mathrm{IB}_{\mathrm{shift}}$ reduces to

$$\mathrm{IB}_{\mathrm{shift}} = -\frac{w_0}{\Phi_0}(0.508\ \overline{\mathrm{ETM}} + 0.767\ \overline{\mathrm{ITM}} + 0.542\ \mathrm{RM}) \tag{4}$$

where Φ_0 is the divergence angle of an arm cavity and w_0 is the waist size of an arm cavity,

$$\Phi_0 = 9.63\times10^{-6}\ \mathrm{rad} \quad \text{and} \quad w_0 = 3.50\times10^{-2}\ \mathrm{m}. \tag{5}$$

4 LIGHT LEVELS AT THE EXTRACTION PORTS

The light intensities at the dark port, in reflection, inside the recycling cavity, inside the arm cavities and in transmission through the end test masses are listed in Table 5 — each for the carrier, the resonant sidebands and the non-resonant sidebands. A perfect contrast was assumed and no losses, except the transmission through the ETMs.

Table 5: Light Intensities. All numbers are in Watts assuming a 10 W input beam.

		port		RC	arm cavity	
Light	**input**	**dark[1]**	**reflect.**	**inside**	**inside**	**trans.[2]**
carrier	8.8	0	0.41	32	21k	4.2
resonant sideband	1.2	1.2	0.02	37	0.14	$< 10^{-4}$
NR sideband	0.012	$< 10^{-5}$	0.012	$< 10^{-3}$	$< 10^{-6}$	$< 10^{-9}$

[1] Perfect contrast is assumed.

[2] These numbers are not realistic, since the ETM transmission has to account for all losses in the arm cavity. In reality, these values are at least an order of magnitude lower.

5 SHOT NOISE LIMITS FOR WAVEFRONT SENSING SIGNALS

Knowing the light levels at the extraction ports one is now able to calculate the shot noise which limits the wavefront sensing. Following the derivation in section 2.2.3 of ref. [4] and using eqn. (1), the shot noise limited detection angles can be written as:

$$\Theta_{SN} = \frac{1}{|A|}\sqrt{\frac{q\,(P_i/P)}{\varepsilon P J_0(\Gamma) J_1(\Gamma)\, f_{\text{split}}}} \tag{6}$$

where q is the elementary charge, P_i the light level at the extraction port, ε the photodetector efficiency and $|A|$ is the absolute value of the signal amplitude for the corresponding wavefront sensor:

$$|A| = \sqrt{\sum_i A_i^2} \tag{7}$$

The shot noise limited detection angles are listed in Table 6, assuming that the contrast of the interferometer is perfect, that the fraction of light split-off for each wavefront sensor is $f_{\text{split}} = 10^{-3}$ at each extraction port, that the input laser intensity is $P = 10\ \text{W}$ and that the photodetector efficiency for YAG laser light is $\varepsilon = 0.35\ \text{A/W}$. It can be seen from Table 6 that the shot noise limited detection angles are tiny compared to the required angular control as derived in ref. [5].

Table 6: Shot noise limited detection angles. The values are given in units of the arm cavity divergence angle per $\sqrt{\text{Hz}}$.

wavefront sensors				
1	**2**	**3**	**4**	**5**
1.0×10^{-10}	6.1×10^{-10}	0.63×10^{-10}	5.9×10^{-10}	7.1×10^{-10}

6 CENTERING

Centering of a laser beam relative to the CoG (center of gravity) of a mirror is important, since the coupling of the thermal noise for yaw and pitch of a mirror into the gravitational wave signal is directly proportional to the centering error. The center of a laser beam can be measured by determining the CoG of a beam profile, for example, with a CCD camera. Table 7 lists the effects of a angular misalignment on the center positions of the laser beam at the dark port, in reflection, inside the recycling cavity on the recycling mirror and inside the arm cavities on the test masses.

Table 7: Matrix of beam center sensitivity. All numbers are in units of waist size per divergence angle of the arm cavity.

	angular degree-of-freedom							
mirror	**ΔETM**	**ΔITM**	**ETM**	**ITM**	**RM**	**BS**	**$\mathrm{IB_{tilt}}$**	**$\mathrm{IB_{shift}}$**
dark port	$< 10^{-4}$	$< 10^{-2}$	0.338	–20.2	28.4	28.9	–0.517	0.940
reflected	$< 10^{-4}$	$< 10^{-3}$	6.36	–5.63	11.9	12.1	–0.217	–4.34
recycling	$< 10^{-4}$	$< 10^{-4}$	1.10	–1.63	2.96	3.02	–0.054	0.083
ITM1	–1.16	–0.527	1.17	0.532	$< 10^{-2}$	$< 10^{-3}$	$< 10^{-3}$	$< 10^{-3}$
ITM2	1.16	0.527	1.17	0.532	$< 10^{-2}$	0.011	$< 10^{-3}$	$< 10^{-3}$
ETM1	–0.845	–1.16	0.845	1.16	0.013	$< 10^{-3}$	$< 10^{-3}$	$< 10^{-3}$
ETM2	0.845	1.16	0.845	1.16	0.013	0.026	$< 10^{-3}$	$< 10^{-3}$

During detection mode the angles of the recycling mirror and the test masses are controlled by the wavefront alignment system, leaving only the angle of the beamsplitter and the tilt and shift of the input beam as degree-of-freedoms to be available for a centering control system. Assuming that the wavefront alignment system acts as a null servo on its degree-of-freedoms, a misalignment of the beamsplitter or a tilt and shift of the input beam can be measured by determining the centers of the beams in reflection of the interferometer and in transmission through the end test masses.

The wavefront sensor signals in matrix form read:

$$\overrightarrow{WFS} = \boldsymbol{A}\Theta + \boldsymbol{B}\Psi \qquad (8)$$

where $\boldsymbol{A}$ is the alignment matrix, Θ is a 5 component vector of the misalignment angles of recycling mirror and test masses, $\boldsymbol{B}$ is a 5×3 matrix describing the wavefront sensing signals coming from the beamsplitter and the input beam and Ψ is a 3 component vector of the misalignment angles of the beamsplitter and the tilt and shift of the input beam.

$$\Psi = \begin{bmatrix} \mathrm{BS} \\ \mathrm{IB_{tilt}} \\ \mathrm{IB_{shift}} \end{bmatrix} \qquad (9)$$

The null-servo condition for the wavefront alignment system takes then the simple form

$$\overrightarrow{WFS} = 0 \quad \text{or} \quad \Theta = -\boldsymbol{A}^{-1}\boldsymbol{B}\Psi \tag{10}$$

If the centering signals δ_x for the reflected light and the light transmitted through the end test masses are written analogously

$$\delta_x = \boldsymbol{C}\Theta + \boldsymbol{D}\Psi \tag{11}$$

with $\boldsymbol{C}$ a 3×5 and $\boldsymbol{D}$ a 3×3 matrix, one can use eqn. (10) to eliminate Θ

$$\delta_x = (-\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{B} + \boldsymbol{D})\Psi \equiv \boldsymbol{Z}\Psi \tag{12}$$

The values for the centering matrix $\boldsymbol{Z}$ are given in Table 8.

Table 8: Centering Matrix. The values are in units of arm cavity waist size per arm cavity divergence angle (for BS and IB_{tilt}) or per arm cavity waist size (IB_{shift}).

port	BS	IB_{tilt}	IB_{shift}
reflection	–0.033	–0.011	1.02
transmission ETM_1	0.019	–1.12	1.33
transmission ETM_2	2.46	–1.12	1.33

7 GW-SIGNAL DUE TO MISALIGNMENT

In first order, an angular misalignment of the interferometer does not produce a signal at the dark port which could be interpreted as a gravitational wave signal. But, in second order, a misalignment of one mirror can generated TEM_{10} mode which is then transformed back into TEM_{00} by an angular misalignment of a second mirror. This TEM_{00} might then leave the interferometer through the dark port and imitate a gravitational wave signal. For the perfectly aligned interferometer the gravitational-wave read-out at the dark port is, of course, zero. We can then write the most general second order equation for a gravitational wave signal GWS_{angle} due to angular misalignment as follows:

$$GWS_{angle}(\Theta) = \frac{1}{2}\Theta H \Theta \tag{13}$$

where Θ is a 5(10) component vector of the mirror misalignment angles and H is the Hessian matrix. The matrix H can be diagonalized; its eigenvalues and eigenvectors are given in Table 9. For obtaining these values it was assumed that the rotation points (axes) of the mirror misalignments are lying in the center of the beam at the reflecting surfaces of the mirrors. In reality, the CoG of a mirror lies about 5 cm behind the surface and the laser beam might hit the mirror off center. The effect coming from the CoG not lying at the surface adds terms of the form $h_i\Theta_i^2/2$ for each mirror misalignment. If the angles are measured in units of the arm cavity divergence angle, the absolute value for h_i is of the order 5×10^{-12} m of differential arm length change and, hence, much smaller than the effect calculated in eqn. (13).

Table 9: Eigenvalues and eigenvectors of the Hessian matrix describing the gravitational wave signal due to angular misalignment. The eigenvalues are given in units of meters of differential arm length change (added to the on-line arm and subtracted from off-line arm length) times the square of the arm cavity divergence angle.

eigenvalues	eigenvectors				
	ΔETM	ΔITM	ETM	ITM	RM
4.74×10^{-7}	**0.552**	**0.442**	**–0.491**	**–0.502**	0.084
-4.74×10^{-7}	**0.552**	**0.442**	**0.491**	**0.502**	–0.084
1.24×10^{-7}	**0.442**	**–0.552**	**0.509**	**–0484**	0.082
-1.24×10^{-7}	**0.442**	**–0.552**	**–0.509**	**0484**	0.082
2.9×10^{-12}	0	0	–0.001	0.166	**0.986**

If we require that the gravitational wave signal at about 150 Hz due to angular misalignment is smaller than $5{\times}10^{-21}\ \mathrm{m}/\sqrt{\mathrm{Hz}}$ and if we assume that the alignment servo system controls the angles within 10^{-3} of a divergence angle at small frequencies (<10 Hz), then the angular jitter at 150 Hz has to be smaller than $2{\times}10^{-11}/\sqrt{\mathrm{Hz}}$ of a divergence angle.

8 INPUT BEAM JITTER

Input beam jitter can couple directly into the gravitational wave read-out, if the interferometer is misaligned [6]. This effect is very small for a common misalignment of the test masses and the recycling mirror. It is more pronounced for the differential misalignment of the test masses. One can then write the induced gravitational wave signal at the dark port as:

$$GWS_{jitter}(\alpha, \Delta x, \Theta) \propto \frac{1}{2}\sum_i \Theta_i(B_i\alpha + C_i\Delta x) \tag{14}$$

Table 10 lists the equivalent differential arm length change for the constants B_i and C_i in units of divergence angle of the misalignment angle and of divergence angle or of waist size of the input beam tilt or shift, respectively. Alternatively, one can also look at second order effects from the beam jitter alone. These effects are, however, completely negligible.

If we require that the gravitational wave signal at about 150 Hz due to beam jitter is smaller than $5{\times}10^{-21}\ \mathrm{m}/\sqrt{\mathrm{Hz}}$ and if we assume that the alignment servo system controls the angular degree-of-freedoms to within 10^{-3} of a divergence angle at small frequencies (< 10Hz), then the beam jitter at 150 Hz has to be smaller than $3{\times}10^{-9}/\sqrt{\mathrm{Hz}}$ of a divergence angle for both tilt directions and smaller than $2{\times}10^{-8}/\sqrt{\mathrm{Hz}}$ of a waist size for the shifts in x and y.

Table 10: Gravitational-wave sensitivity to beam jitter and misalignment. The values are in units of meters of differential arm length change per arm cavity divergence angle and per arm cavity divergence angle (for B_i) or arm cavity waist size (for C_i); see also ref. [6].

	angular degree-of-freedom				
input beam	**ΔETM**	**ΔITM**	**ETM**	**ITM**	**RM**
tilt (B_i)	**-19.4×10^{-10}**	**-8.77×10^{-10}**	$< 10^{-13}$	$< 10^{-13}$	$< 10^{-12}$
shift (C_i)	**-3.02×10^{-10}**	**-1.49×10^{-10}**	$< 10^{-13}$	$< 10^{-13}$	$< 10^{-13}$

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