LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY
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Modal Model Update 2

GW-Sensitivity to Angular Misalignments

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1 Abstract

This document describes the calculation and presents the results for the change of gravitational-wave sensitivity at the dark port of the LIGO interferometer for angular misalignments. It is found that the most sensitive angular misalignment is due to a common misalignment of the ITMs (input test masses) and an opposite misalignment of twice the size of the recycling cavity mirror. Requiring that the loss of the gravitational-wave signal-to-noise sensitivity at the dark port does not exceed 0.5% of its maximum value for a perfectly aligned interferometer sets an upper limit of the average r.m.s. angular misalignment for each mirror of 0.8×10^{-8} rad.

2 Definitions

If $S(\Delta L)$ is the down-converted signal at the dark port as a function of the differential arm cavity length ΔL, the gravitational-wave signal sensitivity can be written as:

$$S_{sens} = \frac{d}{d\Delta L} S(\Delta L) \tag{1}$$

This sensitivity has a maximum, if the interferometer is perfectly aligned. Hence, for a misaligned system, it can be approximated by:

$$S_{sens}(\vec{\varphi}) = S_{sens}(0)\left[1 + \frac{1}{2}\vec{\varphi} H \vec{\varphi}\right] \tag{2}$$

where the $\vec{\varphi}$ is the 5-component vector of the horizontal (vertical) misalignment angles. The matrix H is sometimes called Hessian matrix

$$H_{ij} = \frac{d^2}{d\varphi_i \, d\varphi_j} S_{sens}(\vec{\varphi}) \tag{3}$$

It is up-to a constant the inverse of the covariance matrix C:

$$C = 2H^{-1} \tag{4}$$

Diagonalizing the covariance matrix gives the eigenvectors u_i which are the axes of the variance-ellipsoid (in the 5 dimensional angular space) and the corresponding eigenvalues σ_i^2 which are the square of the axes lengths (variances). Using the new basis u_i to express the misalignment angles ψ_i the relative loss of sensitivity ε can be easily calculated by:

$$\varepsilon = -2\sum_i \left(\frac{\psi_i}{\sigma_i}\right)^2 \tag{5}$$

where the factor of 2 comes from the fact that we have so-far neglected the vertical misalignment angles. If the r.m.s. angular misalignment $\Delta\varphi_{rms}$ are equal for all degree of freedoms, one obtains

$$\Delta\varphi_{rms} = \sqrt{\frac{-\varepsilon}{2\sum_i \frac{1}{\sigma_i^2}}} \tag{6}$$

A completely similar result can be derived for the signal-to-noise ratio of the gravitational-wave detection at the dark port by replacing the signal sensitivity S_{sens} in equation (1) with the signal-to-noise sensitivity:

$$\left(\frac{S}{N}\right)_{sens} = \frac{\frac{d}{d\Delta L}S(\Delta L)}{\sqrt{P_{cr} + \frac{3}{2}P_{sb}}} \tag{7}$$

where P_{cr} and P_{sb} are the average light intensities of the carrier and the sidebands leaking out of the dark port (predominantly stored in the sidebands).

The misalignment angles used for the calculations are linear combinations of the individual mirror angles φ_i:

$$\begin{bmatrix} \Delta ETM \\ \Delta ITM \\ \overline{ETM} \\ \overline{ITM} \\ RM \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \end{bmatrix} \tag{8}$$

where φ_1 and φ_2 are the misalignment angle of the ITM and ETM mirrors of the on-line arm, respectively, φ_3 and φ_4 are the misalignment angle of the ITM and ETM mirrors of the off-line arm, respectively, and φ_5 is the misalignment angle of the recycling mirror (see Ref. [3] for the sign convention of the misalignment angles). All angles are measured in units of the beam divergence angle in the arm cavities Φ_0:

$$\Phi_0 = \frac{w_0}{z_0} \approx 9.65\times10^{-6}\text{rad} \tag{9}$$

with w_0 and z_0 the beam waist size and the Rayleigh length, respectively.

3 Results

The covariance matrix was calculated and diagonalized for three cases:
(*i*) signal-to-noise ratio of the gravitational-wave read-out,
(*ii*) signal strength of the gravitational-wave read-out and
(*iii*) noise level at the gravitational-wave read-out port.

The directions of the ellipsoid axes u_i — together with variances σ_i^2 — are given in Table 1. It can be seen that the most sensitive misalignment is a common rotation of the ITM against an opposite rotation of the recycling mirror. The least sensitive misalignment is the one where all

Table 1: Covarinace Matrix. Eigenvalues (variances) and eigenvectors (axes direction of variance ellipsoid).

	eigenvalue σ_i^2	eigenvector (ellipsoid axis)				
		ΔETM	ΔITM	$\overline{ETM}$	$\overline{ITM}$	RM
signal-to-noise	–6.39	0.000	0.000	**0.393**	**–0.747**	**–0.537**
	–0.834	0.000	–0.000	**0.920**	**0.317**	**0.231**
	–0.116	**–0.421**	**0.909**	0.000	0.000	0.000
	–0.000496	**0.909**	**0.417**	0.000	–0.004	0.005
	–0.000612	–0.005	–0.002	–0.002	**–0.584**	**0.811**
signal	–4.19	0.000	0.000	**–0.487**	**0.711**	**0.507**
	–0.694	0.000	0.000	**0.873**	**0.399**	**0.280**
	–0.0596	**–0.455**	**0.890**	0.000	0.000	0.000
	–0.00545	**0.890**	**0.455**	0.000	0.000	0.000
	–0.000303	0.000	0.000	0.004	**–0.579**	**0.815**
noise	420	0.000	0.000	**0.972**	**0.198**	**0.128**
	–17.5	0.000	0.000	**0.236**	**–0.794**	**–0.560**
	–0.118	**–0.413**	**0.911**	0.000	0.000	0.000
	0.000545	**0.911**	**0.413**	0.000	0.000	0.000
	–0.000602	0.000	0.000	0.010	**–0.574**	**0.819**

mirrors are rotated in the same direction. One also notices that the alignment of the ETM mirrors is as critical as the alignment of the ITM and RM mirrors. If the alignment can done equally well for all angular degree of freedoms and if the loss of sensitivity must not exceed 0.5%, each r.m.s. misalignment angle should be smaller than $\Delta\varphi_{rms} = 8.0\times10^{-9}$ rad.

A negative sign of the eigenvalue indicates a maximum, whereas a positive sign indicates a minimum. For the signal and the signal-to-noise ratio all eigenvalues are negative and, hence, the perfectly aligned case is a true maximum of sensitivity. For the noise (square root of light power) this is not true, i.e. that for some misalignments the power at the dark port increases and for others it decreases.

Table 2: Interferometer parameters. LIGO 4 km configuration.

Parameter	Unit	arm (ITM)	arm (ETM)	recycl. (RM)
length (common / differential)	m		3999.01	9.38 / 0.21
power transmission	%	3	0.0015	2.44
power reflectivity	%	96.995	99.9935	97.5
radius of curvature	m	–14571	7400	–9998.65
modulation frequencies	MHz		23.971	35.956
modulation depths	Γ		0.45	0.045
wave length	µm			1.064
refractive index				1.44968

REFERENCE

[1] Y. Hefetz, N. Mavalvala and D. Sigg, *“Principles of Calculating Alignment Signals in Complex Optical Interferometers“*, LIGO-P96024-A-D (1996).

[2] Y. Hefetz and N. Mavalvala, Proc. Seventh Marcel Grossman Meet. on Gen. Rel. (1994).

[3] D. Sigg, “*Modal Model Update 1: Interferometer Operators*”, LIGO-T960113-00-D (1996).

APPENDIX A 2 KM INTERFEROMETER

Table 3: Covarinace Matrix. Eigenvalues (variances) and eigenvectors (axes direction of variance ellipsoid) for the 2 km system.

	eigenvalue σ_i^2	eigenvector (ellipsoid axis)				
		ΔETM	ΔITM	$\overline{ETM}$	$\overline{ITM}$	RM
signal-to-noise	–7.88	0.000	0.000	**0.538**	**–0.685**	**–0.492**
	–0.499	0.002	–0.003	**0.843**	**0.434**	**0.318**
	–0.205	**–0.587**	**0.809**	0.003	0.002	0.000
	–0.000301	**0.809**	**0.587**	0.000	0.000	0.002
	–0.000519	–0.002	0.000	–0.004	**–0.586**	**0.811**
signal	–5.25	0.000	0.000	**–0.591**	**0.657**	**0.468**
	–0.434	0.000	0.000	**0.807**	**0.482**	**0.342**
	–0.0849	**–0.610**	**0.793**	0.000	0.000	0.000
	–0.00329	**0.793**	**0.610**	0.000	0.000	0.000
	–0.000257	0.000	0.000	0.001	**–0.580**	**0.815**
noise	188	0.000	0.000	**0.914**	**0.335**	**0.228**
	–32.3	0.000	0.000	**0.408**	**–0.747**	**–0.527**
	–0.140	**–0.585**	**0.811**	0.000	0.000	0.000
	0.000331	**0.811**	**0.585**	0.000	0.000	0.000
	–0.000511	0.000	0.000	0.006	**–0.574**	**0.819**

Table 4: Interferometer parameters. LIGO 2 km configuration.

Parameter	Unit	arm (ITM)	arm (ETM)	recycl. (RM)
length (common / differential)	m	2003.35		11.67 / 0.26
power transmission	%	3	0.0015	2.44
power reflectivity	%	96.995	99.9935	97.5
radius of curvature	m	–14571	7400	–9935.74
modulation frequencies	MHz	19.267		28.900
modulation depths	Γ	0.45		0.045
wave length	μm	1.064		
refractive index		1.44968		