New Folder Name

Frequency Response Derivation

Frequency response derivation, MR, Draft 1, 1/13/1994

Fred: Please have a look at this with a critical eye towards organization and level of detail; please ignore spelling mistakes or examples of awkward wording since major revisions will likely be required anyway. — MR.

1 Introduction

Consider a Fabry-Perot interferometer, consisting of two mirrors as shown in Fig xxx. It is often desirable to maintain the length of such an interferometer such that a beam of light from a laser resonates in the interferometer. In the presence of noise from the environment, such as acoustic motion of the cavity mirrors or laser frequency noise, this requires a control system capable of detecting the deviations from resonance of the interferometer and of exerting an opposing influence on the length of the interferometer. For the sake of designing such a control system, it is necessary to know the transfer function of the system to be controlled, sometimes called the *plant.* Suppose that the laser light is phase-modulated and that a photodiode and demodulator are used as shown to detect deviations from resonance. Now suppose we cause the mirror to move such that its position is given by:

$$\mathrm{x}(t) = Re\{xe^{st}\}$$

then for sufficiently small x the output of the demodulator will be given by

$$\begin{aligned}\mathrm{v}(t) &= Re\{ve^{st}\} \\ &= Re\{H(s)xe^{st}\}\end{aligned}$$

This is our definition of $H(s)$, the *transfer function* of the plant. If one is designing a control system for a more complicated interferometer, the basic issue is the same: one needs to know the transfer functions from the available inputs to the available outputs.

We will derive these transfer functions in three stages. First we will show that a shaking mirror acts as a source of light at frequencies different from the frequencies of light incident on it. Then we will derive the relation between the spectral content of the light incident on a photodiode and the output of a demodulator following that photodiode. Finally we will derive the frequency dependence of the throughput from the sources of light (the shaking mirrors) to the interferometer outputs.

2 Effect of a shaking mirror

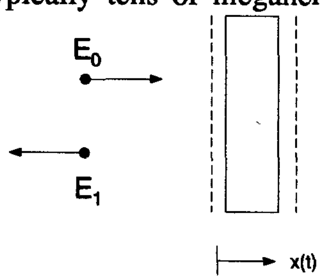
Consider a section of one of the laser beams within our interferometer. It contains electromagnetic radiation travelling in two directions, which we will call the z and —z directions:

$$\vec{\mathrm{E}}(z,t) = \vec{\mathrm{E}_0} Re\left\{E_r(t) e^{i(\nu t - kz)} + E_l(t) e^{i(\nu t + kz)}\right\}$$

We will assume uniform linear polarization throughout, and concern ourselves only with the complex functions of time $E_r(t)$ and $E_l(t)$, which we will refer to as the *fields* traveling in the z and –z directions.

Correct E_0, E_1 etc! to E_l, E_r

Now consider a mirror which is shaking sinusoidally at an "audio" frequency $\omega/2\pi$ as shown in Figure 1 (we use the term "audio" to denote frequencies lower than 100 kHz, and therefore much lower than the modulation frequency, which is typically tens of megahertz).



Its position is given by

$$\mathrm{x}(t) = Re\{X e^{i\omega t}\}$$

We assume that the amplitude of the motion is very small ($|kX| \ll 1$). This assumption is in fact necessary for our transfer function to be well defined. If the amplitude were large, then there would not be a linear relationship between the mirror position and the demodulator voltage. The motion of the mirror modulates the path length between the incident and reflected field, so that

$$\begin{aligned} E_1 &= \sqrt{R_1} E_0 e^{-i2k\mathrm{x}(t)} \\ &\simeq \sqrt{R_1} E_0 \left[J_0(2kX) + J_1(2kX)e^{i\omega t} + J_{-1}(2kX)e^{-i\omega t}\right] \\ &\simeq \sqrt{R_1} E_0 \left[1 + kXe^{i\omega t} - kXe^{-i\omega t}\right] \end{aligned}$$

state $J_1(\Gamma) \simeq \frac{\Gamma}{2}$ for $\Gamma \ll 1$, $J_0(\Gamma) \simeq 1$

Thus for a single frequency of light incident on the mirror, the reflected beam contains three frequencies, the ("carrier") frequency of the incident beam and two audio sidebands. If the incident beam were already to contain audio sideband fields, then in principle each of these would result in two additional audio sidebands, two of which would fall back onto the carrier. However all four of these additional sidebands are of order $(kX)^2$ so we neglect them. The result then, is that the newly generated audio sidebands on the carrier add to those already present.

If there are RF sidebands present in the incident light, then each will produce a pair of audio sidebands in the reflected light.

3 RF modulation and demodulation

We stated above that the light entering the interferometer is phase modulated.

$$\begin{aligned} E_{incident} &= E_0 e^{i\Gamma \sin \Omega t} \\ &= J_0(\Gamma) + J_1(\Gamma)e^{i\Omega t} - J_1(\Gamma)e^{i-\Omega t} + \cdots \end{aligned}$$

The above Fourier series contains an infinite number of terms, but for any finite modulation index there exists some order N such that we can neglect terms of order exceeding N *Find justification in terms of asymptotic expression for large orders in A and S* Then we write

$$E_{incident} \simeq J_0(\Gamma) + J_1(\Gamma)e^{i\Omega t} - J_1(\Gamma)e^{-i\Omega t} + \cdots + J_N(\Gamma)e^{iN\Omega t} - J_N(\Gamma)e^{-iN\Omega t}$$

Need to re-check correctness of sign in above, especially for N terms.

We will consider two types of demodulator: an "inphase" demodulator and a "quadrature phase" demodulator. These are of course the same kind of device (typically a double-balanced mixer), but the former is driven (at its local oscillator input) by the phase-modulating signal, the latter by the same signal delayed ninety degrees. We will model the effect of these devices as follows:

mention that $V_P \propto |E_{photodetector}|^2$

$$V_{D\,in}(t) = \frac{1}{T}\int_{t-T} V_P(t')\cos(\Omega t')dt'$$

$$V_{D\,qu}(t) = \frac{1}{T}\int_{t-T}^{t} V_P(t')\sin(\Omega t')dt'$$

First we will show that the only frequency components of relevance in the demodulator inputs are those near the modulation frequency. The input voltages to the demodulators consist of components at harmonics of the modulation frequency:

write as Σ

$$V_P = f_0(t) + f_{i1}(t)\cos\Omega t + f_{q1}(t)\sin\Omega t + f_{i2}(t)\cos 2\Omega t + f_{q2}(t)\sin 2\Omega t + \cdots$$

where the $f_i(t)$ are slowly-varying compared to the time-scale T, and we can move them out of the integrals. Then

$$\begin{aligned} V_{D\,in}(t) = {} & \frac{f_0(t)}{\Omega T}(\sin\Omega t - \sin\Omega(T-t)) + \frac{f_{i1}(t)}{2} + \frac{f_{i1}(t)}{4\Omega T}[\sin 2\Omega t - \sin 2\Omega(t-T)] \\ & + \frac{1}{2\Omega T}(\sin^2\Omega t - \sin^2\Omega(t-T)) \\ & + \sum_{n=2}^{\infty} \frac{f_{in}(t)}{(2n-1)\Omega T}[\sin((n-1)\Omega t) - \sin((n-1)\Omega(t-T))] \\ & + \sum_{n=2}^{\infty} \frac{f_{in}(t)}{(2n+1)\Omega T}[\sin((n+1)\Omega t) - \sin((n+1)\Omega(t-T))] \\ & + \sum_{n=2}^{\infty} \frac{f_{in}(t)}{(2n-1)\Omega T}[\cos((n-1)\Omega t) - \cos((n-1)\Omega(t-T))] \\ & - \sum_{n=2}^{\infty} \frac{f_{in}(t)}{(2n+1)\Omega T}[\cos((n+1)\Omega t) - \cos((n+1)\Omega(t-T))] \end{aligned}$$

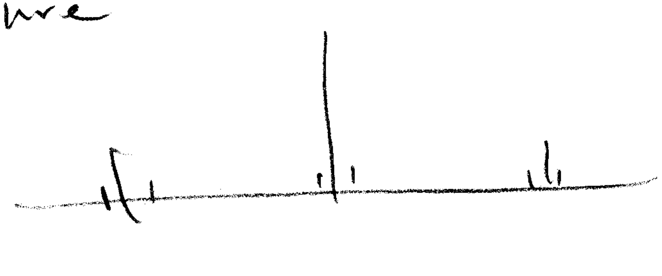
The only term in $V_{D\,in}(t)$ not proportional to $\frac{1}{\Omega T}$ is the second. It is proportional to the component of the input voltage which is at the same frequency and in the same phase as the modulation voltage. In similar fashion, one can show that the only term in $V_{D\,qu}(t)$ not proportional to $\frac{1}{\Omega T}$ is $\frac{f_{q1}(t)}{2}$. ← Sentence to remind reader of origin of frequencies

The field in the light from any output of the interferometer then will contain $6N+3$ frequency components, and the photocurrent is proportional to the square of the modulus of this field:

$$V_P = \left| \sum_{l=-N}^{N} \sum_{m=-1}^{1} A_{lm} e^{i(l\Omega + m\omega)t} \right|^2$$

When expanded, this gives a sum containing $\frac{(6N+3)^2+(6N+3)}{2}$ terms of the form $Re\left\{A^*_{nm} A_{pq} e^{i((n-p)\Omega + (m-q)\omega)}\right\}$ of which we are only interested in those with

Figure



$n - p = \mp 1$ and $m - q = \mp 1$:

$$V_p^1 = \sum_{l=-N}^{N-1} 2Re\{A^*_{l\,-1} A_{l+1\,0} e^{i(\Omega+\omega)t}$$
$$+A^*_{l\,0} A_{l+1\,1} e^{i(\Omega+\omega)t}$$
$$+A^*_{l\,0} A_{l+1\,-1} e^{i(\Omega-\omega)t}$$
$$+A^*_{l\,1} A_{l+1\,0} e^{i(\Omega-\omega)t}\}$$

*need to fill in some detail here*If we define V_i and V_q to be complex numbers such that

$$V_{D\,in}(t) = Re\left\{V_i e^{i\omega t}\right\}$$
$$V_{D\,qu}(t) = Re\left\{V_q e^{i\omega t}\right\}$$

Then

$$V_i = \sum_{l=-N}^{N-1} \left(A^*_{l\,-1} A_{l+1\,0} + A^*_{l\,0} A_{l+1\,1} + A_{l\,0} A^*_{l+1\,-1} + A_{l\,1} A^*_{l+1\,0}\right)$$

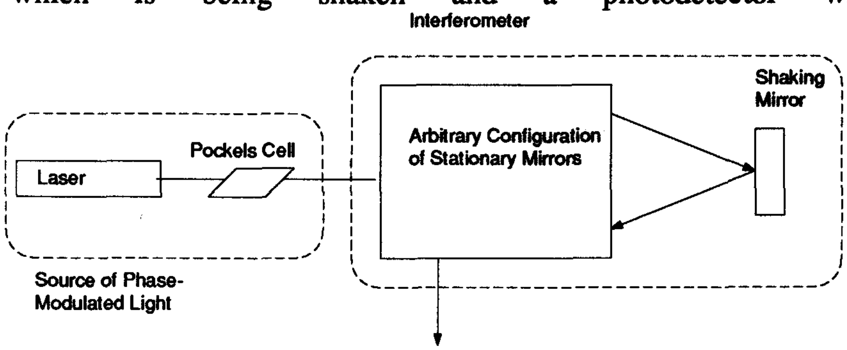
and

$$V_q = \sum_{l=-N}^{N-1} i\left(A^*_{l\,-1} A_{l+1\,0} + A^*_{l\,0} A_{l+1\,1} - A_{l\,0} A^*_{l+1\,-1} - A_{l\,1} A^*_{l+1\,0}\right)$$

4 Transfer Function

Let us return now to the original question: what is the transfer function from mirror displacement to demodulator voltage? We will show that this transfer function is proportional to a sum of optical transfer functions corresponding to transmission from a summing input at the mirror being shaken to the optical output. we will derive these results assuming $N = 1$

Consider a system consisting of a source of phase-modulated light, an interferometer having two inputs and two outputs, a mirror which is being shaken and a photodetector with



demodulators.

We will define six fields and three transfer functions. Let $E_{M\,0}$, $E_{M\,-1}$ and $E_{M\,1}$ be the carrier, lower RF sideband and upper RF sideband fields incident on the shaking mirror. Similarly, let $E_{T\,0}$, $E_{T\,-1}$ and $E_{T\,1}$ be the carrier and RF fields at the interferometer output. Let $U_0(\omega)$, $U_{-1}(\omega)$ and $U_1(\omega)$ be the transfer functions centered on the carrier, lower and upper RF sideband respectively for transmission from summing in at the shaking mirror to the optical output. In particular, $U_0(\omega)$ for example, is the complex ratio of the audio sideband fields near the carrier leaving the output to the fields summed in at the mirror because it is shaking. For mirror motion at angular frequency ω_0, the upper audio sideband of the carrier at the output is related to the amplitude of this sideband summed in at the mirror by $U_0(\omega_0)$; the transmission of the lower audio sideband is proportional to $U_0(-\omega_0)$.

Then the fields at the output are:

$$\sum_{l=-1}^{1} E_{T\,l} e^{il\Omega} + \sum_{l=-1}^{1} E_{M\,l}\Big(kXU_l(\omega)e^{i(\omega+l\Omega)t} - kXU_l(-\omega)e^{i(\omega-l\Omega)t}\Big)$$

and from *equation with A's in it*

$$\begin{aligned} V_i = &-(E_{M\,-1}kXU_{-1}(-\omega))^* E_{T\,0} + E^*_{T\,-1}E_{M\,0}kXU_0(\omega) \\ &- E_{T\,-1}(E_{M\,0}kXU_0(-\omega))^* + E_{M\,-1}kXU_{-1}(\omega)E^*_{T\,0} \\ &- (E_{M\,0}kXU_0(-\omega))^* E_{T\,1} + E^*_{T\,0}E_{M\,1}kXU_1(\omega) \\ &- E_{T\,0}(E_{M\,1}kXU_1(-\omega))^* + E_{M\,0}kXU_0(\omega)E^*_{T\,1} \end{aligned}$$

We see that V_i is of the form

$$V_i = \sum c_k U_k(\omega) + c^*_k U^*_k(-\omega)$$

In the examples which we will consider in a later section, we will have

$$U_k^*(-\omega) = U_k(\omega)$$

and

$$U_1(\omega) = \text{const.}$$
$$U_{-1}(\omega) = \text{const.}$$

in the frequency range of interest to us. Then

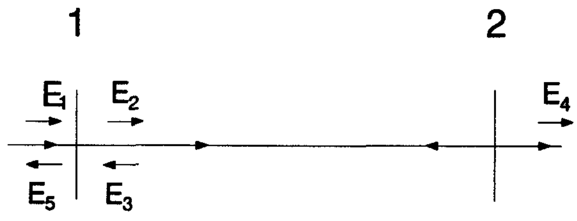
$$H_i = B + DU_0(\omega)$$

The constants B and D can most easily be determined from an analysis of the very-low-frequency behavior of the system.

5 Transmission of Optical Cavities

We will analyze the transmission of two types of optical cavity: a simple two-mirror cavity and a compound or "coupled" three-mirror cavity. The reason the transmission is interesting to us is that (near the carrier and RF sideband frequencies) it is proportional to the transfer functions $U_k(\omega)$ we need. This can be seen from the fact that the summing in of audio sideband light due to the shaking of a mirror is equivalent to the summing in of light incident from outside the interferometer. In both cases the total electric field is the incident field times the reflectivity of the mirror, plus whatever field is being summed in.

Consider first the simple cavity illustrated in Figure xxx.



If the cavity is illuminated from the left with a field of the form

$$E_1 = e^{i\omega t}$$

then we can write

$$
\begin{aligned}
E_2 &= t_1 E_1 - r_1 E_3 \\
E_3 &= r_2 e^{-i(\phi+\omega\tau)} E_2 \\
E_4 &= t_2 e^{-i(\phi+\omega\tau)/2} E_2 \\
E_5 &= r_1 E_1 + t_1 E_3
\end{aligned}
$$

where ϕ is the phase lag accumulated by light at the optical center frequency $\left(2\pi\nu\tau + \text{Gouy}\right)$ ($\omega = 0$) and $\tau = 2l/c$ is the round-trip travel time, with l being the length of the cavity.

Gouy

We have chosen the reflectivity of both mirrors to be negative on the right and positive on the left. Now we solve for E_2 and E_4:

$$
\begin{aligned}
E_2 &= \frac{t_1}{1 + r_1 r_2 e^{-i\phi}} E_1 \\
E_4 &= \frac{t_1 t_2 e^{-i\phi/2}}{1 + r_1 r_2 e^{-i\phi}} E_1
\end{aligned}
$$

If the cavity is resonant at $\omega = 0$ ($\phi = \pi$) and if $\omega\tau \ll 1$ the first expression simplifies to

$$
\begin{aligned}
E_2 &\simeq \frac{t_2}{1 - r_1 r_2 (1 - i\omega\tau)} \\
&= \frac{t_2}{1 - r_1 r_2} \frac{1}{1 + \frac{i\omega}{\frac{1-r_1 r_2}{r_1 r_2 \tau}}}
\end{aligned}
$$

which, if we substitute s for $i\omega$ and define

$$
w_c \equiv \frac{1 - r_1 r_2}{r_1 r_2 \tau}
$$

can be written in the more familiar form:

$$
E_2 \simeq \frac{t_2}{1 - r_1 r_2} \frac{1}{1 + \frac{s}{w_c}}
$$

Here $w_c/2\pi$ is the cavity corner frequency.

Similarly, the transmitted field can be shown to be

$$E_4 = \frac{t_1 t_2}{1 - r_1 r_2} \frac{1 - \frac{s\tau}{2}}{1 + \frac{s}{w_c}}$$

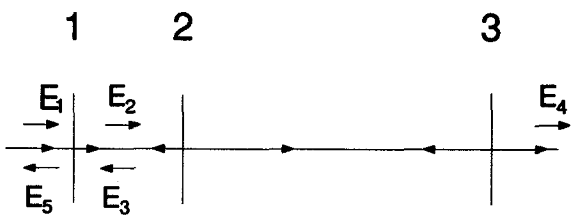
and the reflected field

$$E_5 \simeq -r_{c0} \frac{1 - \frac{s}{\frac{r_{c0}}{r_1} w_c}}{1 + \frac{s}{w_c}}$$

where

$$r_{c0} = \frac{t_1^2 r_2}{1 - r_1 r_2} - r_2$$

Let us proceed now to the case of the coupled cavity shown in Fig xxx



We know from above that

$$E_3 = e^{-i(\phi_1 + w\tau_1)} \left(-r_{c2} \frac{1 - \frac{s}{\frac{r_{c2}}{r_1} w_c}}{1 + \frac{s}{w_c}} \right) E_2$$

where

$$r_{c2} = \frac{t_2^2 r_3}{1 - r_2 r_3} - r_3$$

Now if we define

$$r_c \equiv -r_{c2} \frac{1 - \frac{s}{\frac{r_{c2}}{r_1} w_c}}{1 + \frac{s}{w_c}}$$

and

$$t_c \equiv \frac{t_2 t_3}{1 - r_2 r_3} \frac{1 - \frac{s\tau_2}{2}}{1 + \frac{s}{w_c}}$$

then

$$E_2 = t_1 E_1 - r_1 E_3$$
$$E_3 = r_c e^{i(\phi_1 + \omega\tau_1)} E_2$$
$$E_4 = e^{i(\phi_1 + \omega\tau_1)/2} t_c E_2$$
$$E_5 = r_1 E_1 + t_1 E_3$$

$$
\begin{aligned}
E_2 &= \frac{t_1}{1 + r_1 r_c e^{-i(\phi_1 + \omega\tau_1)}} \\
&= \frac{t_1}{1 - r_1 r_{c2} \frac{1 - \frac{s}{\frac{r_{c2}}{r_2} w_c}}{1 + \frac{s}{w_c}} (1 - s\tau_1)} \\
&= \frac{t_1 \left(1 + \frac{s}{w_c}\right)}{1 + \frac{s}{w_c} - r_1 r_{c2} \left(1 - \frac{r_2}{r_{c2}} \frac{s}{w_c}\right)(1 - s\tau_1)}
\end{aligned}
$$

too many

$$
1 + \frac{s}{w_c} - r_1 r_{c2} \left(1 - \frac{r_2}{r_{c2}} \frac{s}{w_c}\right)(1 - s\tau_1) =
$$