New Folder Name Bandwidth Limitations

Laser-Locking Bandwidth Limitations Imposed by the Response of a Fabry-Perot Cavity

Martin Regehr

March 19, 1991

1 Frequency Response

The response of a Fabry-Perot cavity is given by

$$E_{cav}(t) = t_1^2 r_2 \sum_{n=1}^{\infty} E_{inc}(t - n\tau)(r_1 r_2)^{n-1},$$

where E_{inc} is the field of the incident laser beam, E_{cav} is the field of the beam leaking out of the cavity, τ is the cavity round-trip time, and t_1, r_1, and r_2 are the input mirror amplitude transmission and reflectivity and the output mirror amplitude reflectivity, respectively. If

$$E_{inc} = E_0 e^{-i(2\pi\nu_0 t + \phi(t))}$$

with $\phi(t) \ll 1 \;\; \forall t$, then

$$\begin{aligned} E_{cav} &= t_1^2 r_2 E_0 e^{-i2\pi\nu_0 t} \sum_{n=1}^{\infty} e^{-i\phi(t-n\tau)}(r_1 r_2)^{n-1} \\ &\simeq t_1^2 r_2 E_0 e^{-i2\pi\nu_0 t} \left\{ \frac{1}{1 - r_1 r_2} - i \sum_{n=1}^{\infty} \phi(t - n\tau)(r_1 r_2)^{n-1} \right\} \end{aligned}$$

and

$$\arg\{E_{cav}\} \simeq -2\pi\nu_0 t - (1 - r_1 r_2) \sum_{n=1}^{\infty} \phi(t - n\tau)(r_1 r_2)^{n-1}$$

The voltage V_m at the output of the demodulator is proportional to the product of the magnitudes of the incident and cavity fields and of the phase angle between them. Since $|E_{cav}| \simeq \frac{t_1^2 r_2 |E_0|}{1-r_1 r_2}$, the only factor in this product which depends on $\phi(t)$ is the phase angle between the fields; write

$$V_m(t) \propto \phi(t) - (1 - r_1 r_2) \sum_{n=1}^{\infty} \phi(t - n\tau)(r_1 r_2)^{n-1}$$

This is a real, time-domain equation, so it is possible without ambiguity to introduce a new phasor reference. Let

$$\phi(t) = \phi_0 e^{-i\omega t}$$
$$V_m(t) = V_0 e^{-i\omega t}$$

then

$$\frac{V_0}{\phi_0} = \frac{1 - e^{i\omega\tau}}{1 - r_1 r_2 e^{i\omega\tau}} \qquad 1$$

if ω coincides nearly with a cavity free spectral range, and if $r_1 r_2 \simeq 1$, this transfer function can be approximated by

$$\frac{V_0}{\phi_0} \simeq \frac{-i\delta\tau_s}{1 - i\delta\tau_s} \qquad 2$$

where $\delta \equiv \omega - \frac{2\pi n}{\tau}$ and $\tau_s \equiv \frac{\tau}{1-r_1 r_2}$.

2 Servo Stability

In this section we explore the stability of a servo loop containing the transfer function (1). We assume a cavity 4 km long, with $r_1 r_2 = 0.95$. Bode and Nyquist plots for $\frac{V_0}{\phi_0}$ are shown in Fig. 1 (that the Nyquist plot is exactly a circle can be seen from the fact that (1) is a Möbius transform in the variable $e^{i\omega\tau}$, which, as a function of ω, maps the real line into a circle in the complex plane). The open loop gain, H, of the servo-locked system will be $H(\omega) = \frac{V_0(\omega)}{\phi_0(\omega)} G(\omega)$, where $G(\omega)$ is the gain of the feedback amplifier. A simple feedback amplifier might have $G(\omega) = \frac{K}{-i\omega\tau_s}$; this gives

$$H(\omega) = \frac{K}{-i\omega\tau_s} \frac{1 - e^{i\omega\tau}}{1 - r_1 r_2 e^{i\omega\tau}} \quad .$$

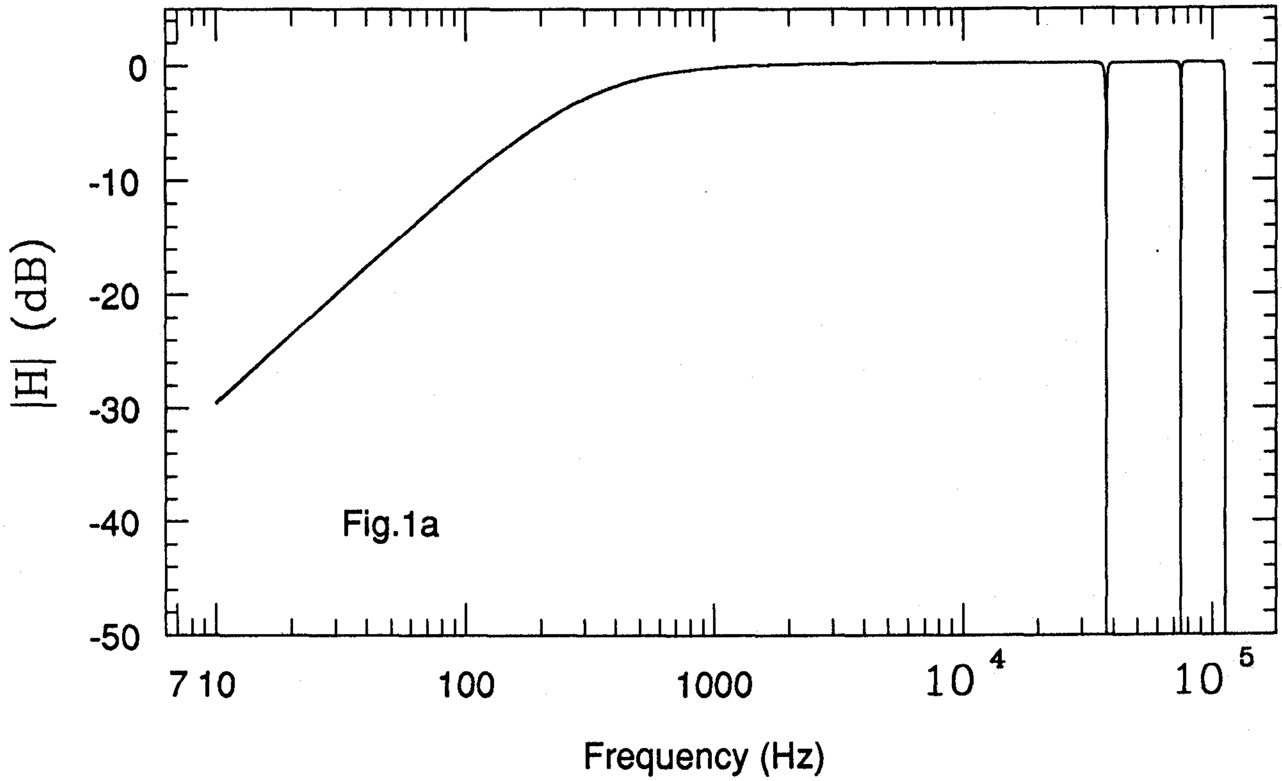
Bode and Nyquist plots for $H(\omega)$ with $K = 200$ are shown in Fig. 2. Clearly the phase margin becomes small near the resonance at the first free spectral range. One solution to this problem is to add pole-zero pairs in order to reduce the phase lag at frequencies above the first resonance. Bode and Nyquist plots for

$$H(\omega) = K \frac{1 + \frac{i\omega}{z_1}}{1 + \frac{i\omega}{p_1}} \frac{1 + \frac{i\omega}{z_2}}{1 + \frac{i\omega}{p_2}} \frac{1}{-i\omega\tau_s} \frac{1 - e^{i\omega\tau}}{1 - r_1 r_2 e^{i\omega\tau}}$$

with $z_1 = (2\pi)\,40\text{kHz}$, $p_1 = (2\pi)\,400\text{kHz}$, $z_2 = (2\pi)\,4\text{MHz}$, $p_2 = (2\pi)\,40\text{MHz}$, and $K = 200$, are shown in Fig. 3 (the curves are traced only up to a frequency of 750 kHz because of the computational expense of tracing the rapidly increasing number of resonances in each octave of increasing frequency). The drawback to this feedback amplifier frequency response is that the slope of its gain above the first resonance is constrained to approximately -10dB per decade and one needs to increase the unity gain frequency considerably for a small increase in gain at the first resonance. The unity gain frequency will probably be limited to a few MHz by technical constraints, and the gain achievable at the first resonance will not be large except for very long cavities. In principle one could also increase the bandwidth past the first resonance by inserting poles and zeroes into the feedback amplifier frequency response to cancel the poles and zeroes given by equation (2). Whether one can generate poles and zeroes with sufficiently high Q's and keep them tuned is uncertain.

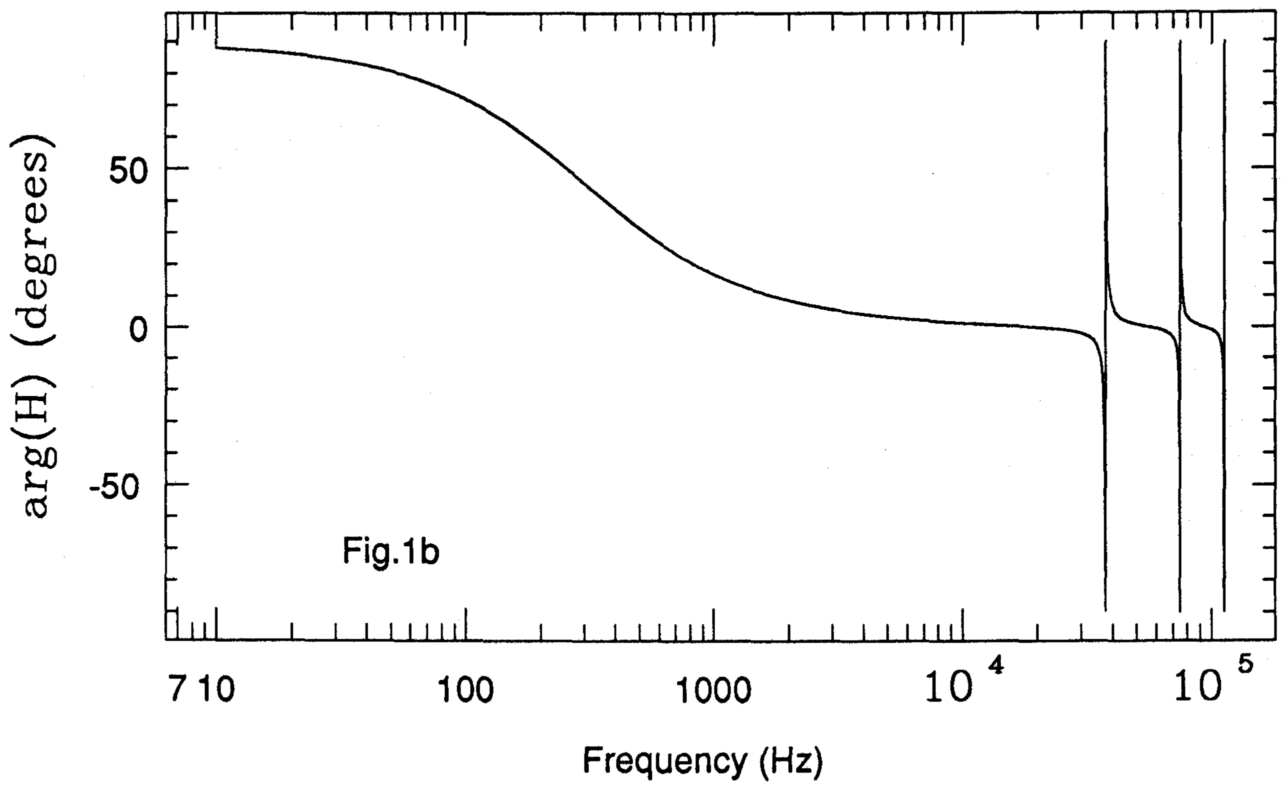
3 Acknowledgements

I would like to thank Alex Abramovici, Lisa Sievers, and Mike Zucker for helpful discussions.



|H| (dB)
0
-10
-20
-30
-40
-50
7 10
100
1000
10^4
10^5
Frequency (Hz)

Fig.1a



arg(H) (degrees)
50
0
-50
7 10
100
1000
10^4
10^5
Frequency (Hz)

Fig.1b

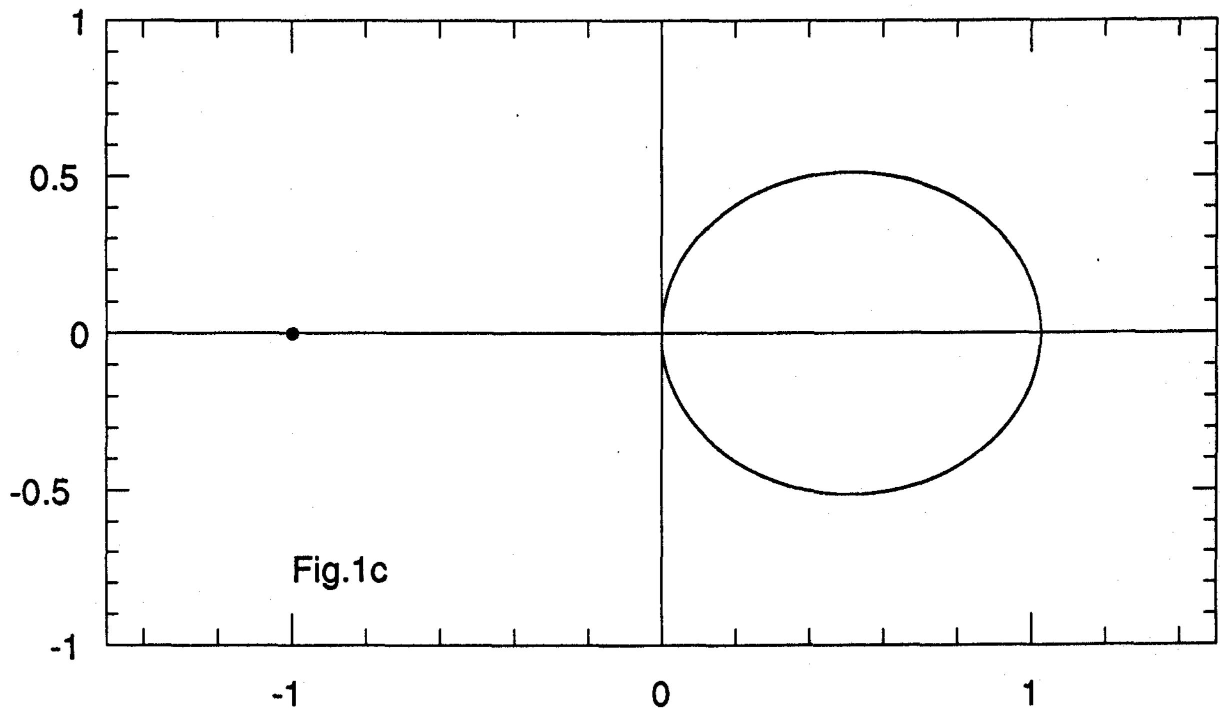

1
0.5
0
-0.5
-1
-1
0
1

Fig.1c

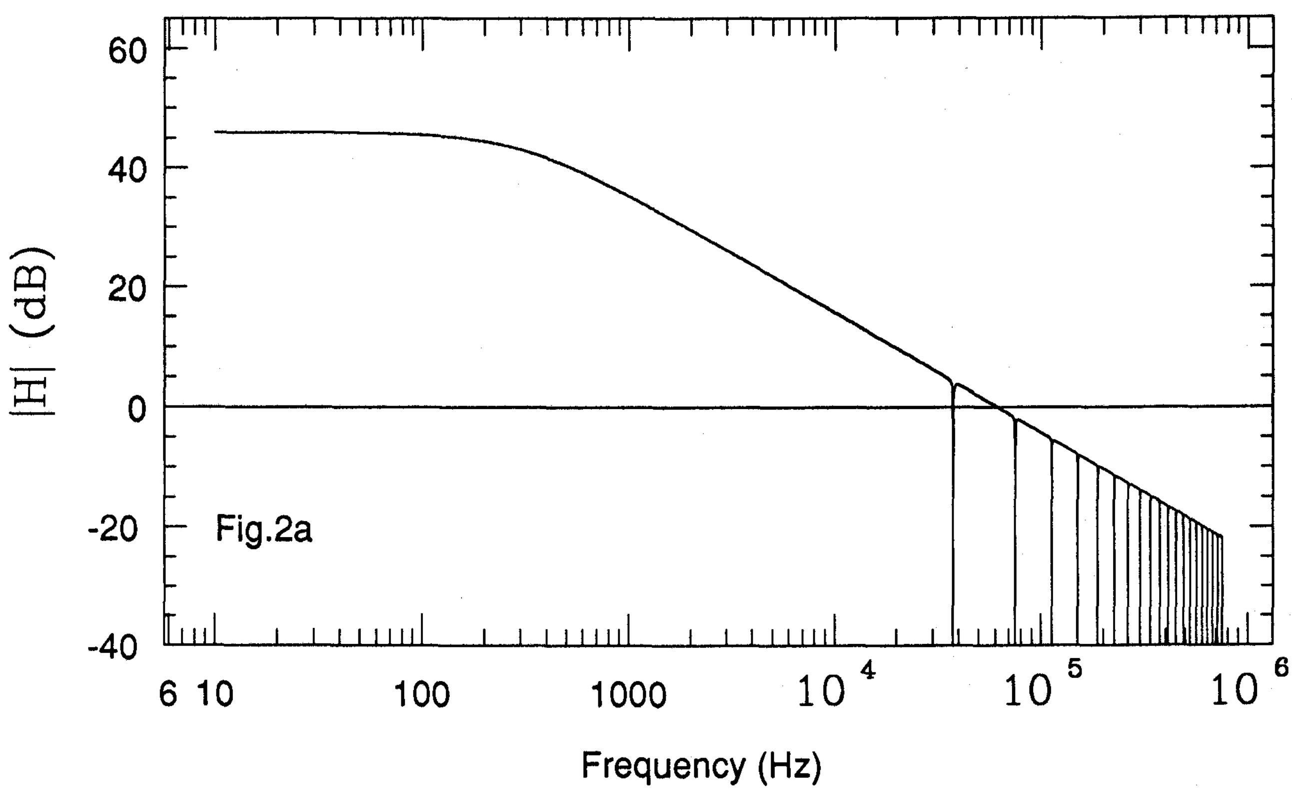
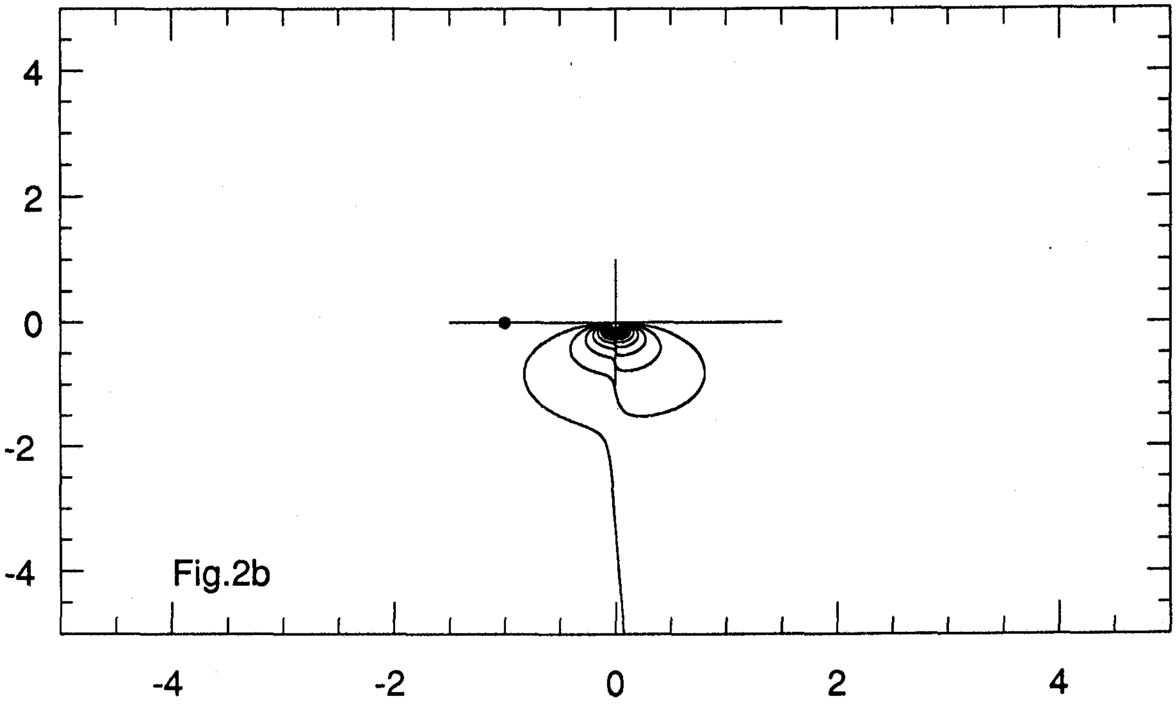
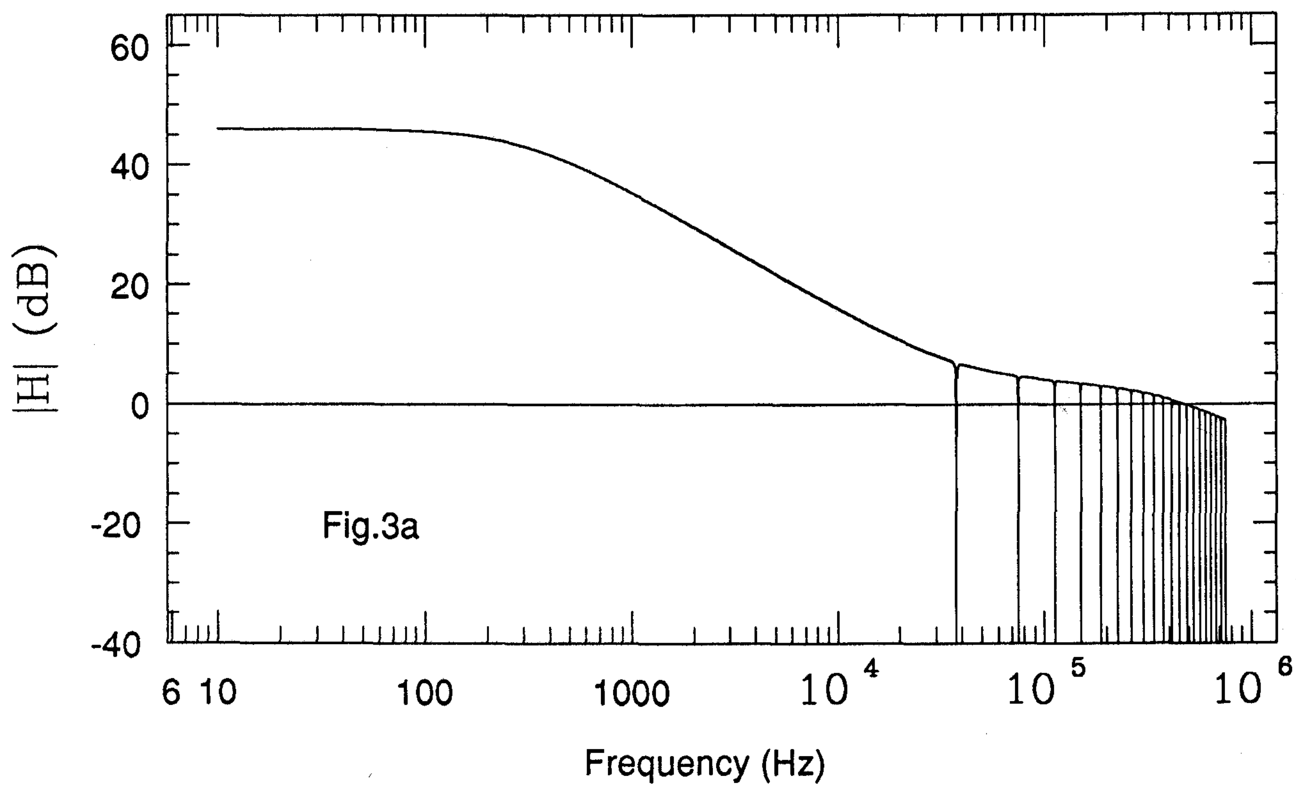

60
40
20
0
-20
-40
|H| (dB)
6 10
100
1000
10 4
10 5
10 6
Frequency (Hz)

Fig.2a


4
2
0
-2
-4
-4
-2
0
2
4
Fig.2b


60
40
20
0
-20
-40
|H| (dB)
6 10
100
1000
10 4
10 5
10 6
Frequency (Hz)
Fig.3a

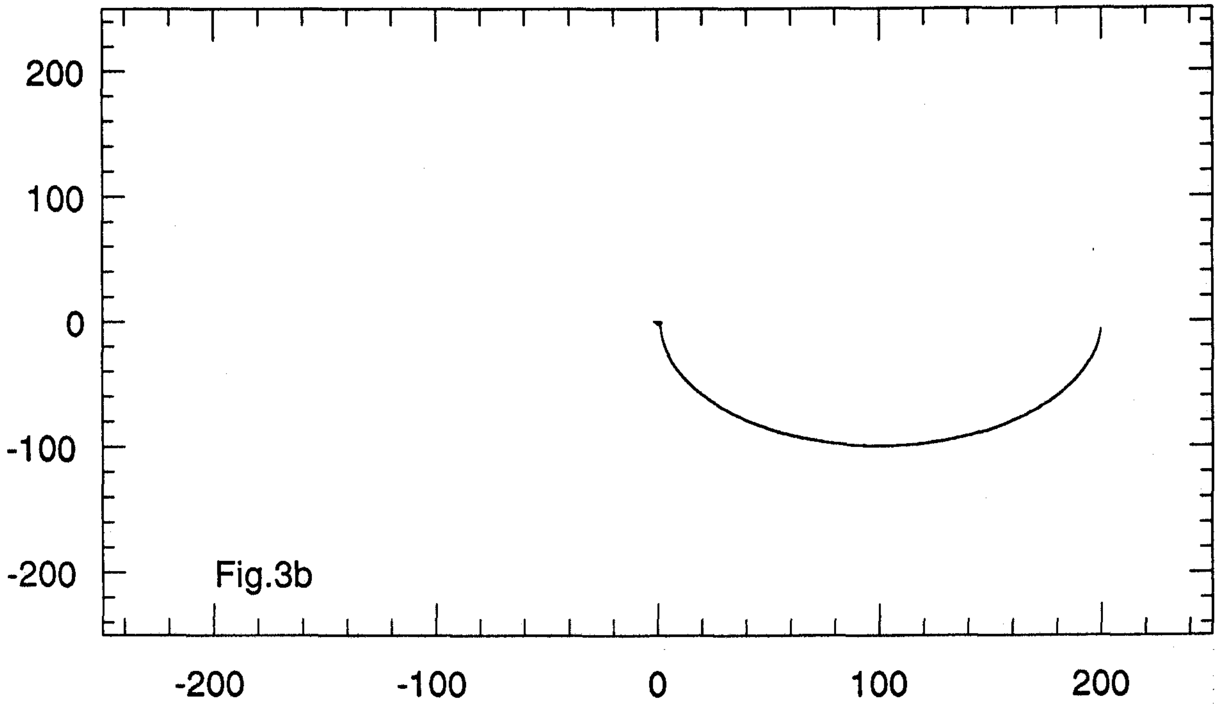


Fig.3b

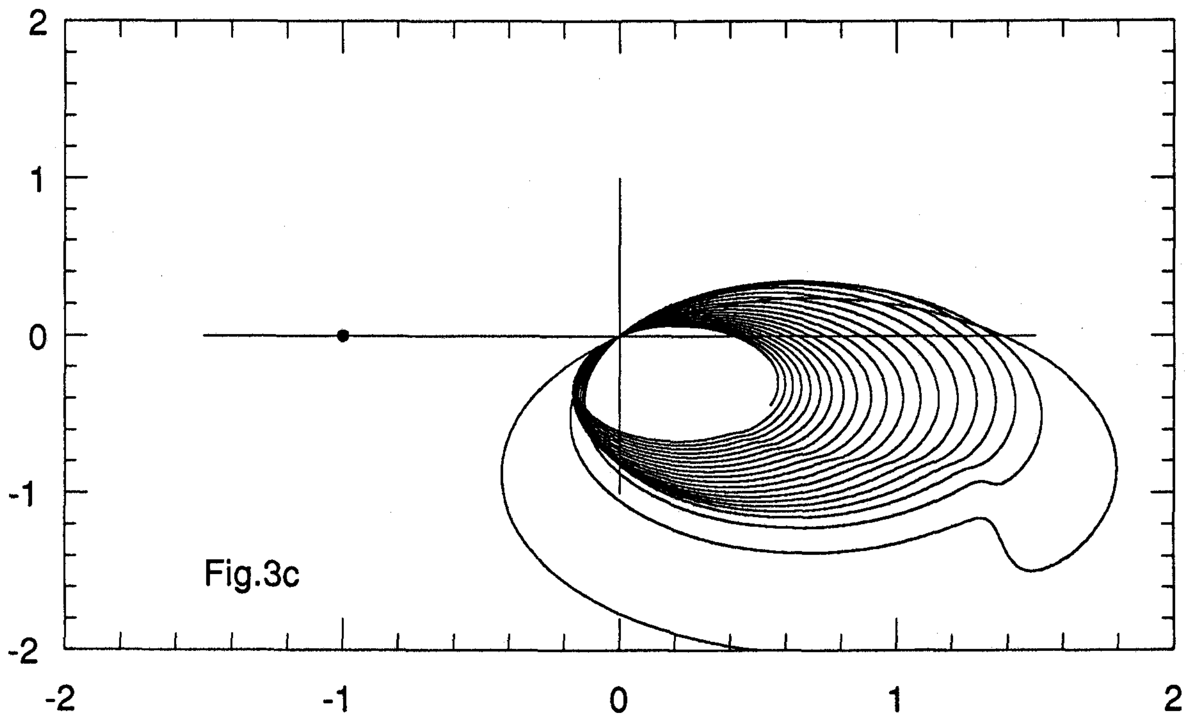


Fig.3c

BATCH
START

STAPLE
OR
DIVIDER

LIGO-T910028-01-D

Laser-Locking Bandwidth Limitations Imposed by the Response of a Fabry-Perot Cavity

Martin Regehr

April 3, 1992

Abstract

The frequency response of a system consisting of a cavity with phase coherent detection of the relative phases of input and leakage fields is derived. A simple feedback arrangement allowing phase-locking of the input light to the cavity with a bandwidth exceeding the free spectral range of the cavity is analyzed. It is found that with this arrangement the slope of the loop gain above the free spectral range frequency of the cavity is limited to about -10 dB/decade.

1 Frequency Response

The response of a Fabry-Perot cavity is given by

$$E_{cav}(t) = t_1^2 r_2 \sum_{n=1}^{\infty} E_{inc}(t - n\tau)(r_1 r_2)^{n-1},$$

where E_{inc} is the field of the incident laser beam, E_{cav} is the field of the beam leaking out of the cavity, τ is the cavity round-trip time, and t_1, r_1, and r_2 are the input mirror amplitude transmission and reflectivity and the output mirror amplitude reflectivity, respectively. If

$$E_{inc} = E_0 e^{-i(2\pi\nu_0 t + \phi(t))}$$

with $\phi(t) \ll 1 \ \ \forall t$, then

$$E_{cav} = t_1^2 r_2 E_0 e^{-i2\pi\nu_0 t} \sum_{n=1}^{\infty} e^{-i\phi(t-n\tau)} (r_1 r_2)^{n-1}$$
$$\simeq t_1^2 r_2 E_0 e^{-i2\pi\nu_0 t} \left\{ \frac{1}{1 - r_1 r_2} - i \sum_{n=1}^{\infty} \phi(t - n\tau)(r_1 r_2)^{n-1} \right\}$$

and

$$\arg\{E_{cav}\} \simeq -2\pi\nu_0 t - (1 - r_1 r_2) \sum_{n=1}^{\infty} \phi(t - n\tau)(r_1 r_2)^{n-1}$$

The voltage V_m at the output of the demodulator is proportional to the product of the magnitudes of the incident and cavity fields and of the phase angle between them. Since $|E_{cav}| \simeq \frac{t_1^2 r_2 |E_0|}{1 - r_1 r_2}$, the only factor in this product which depends on $\phi(t)$ is the phase angle between the fields; write

$$V_m(t) \propto \phi(t) - (1 - r_1 r_2) \sum_{n=1}^{\infty} \phi(t - n\tau)(r_1 r_2)^{n-1}$$

This is a real, time-domain equation, so it is possible without ambiguity to introduce a new phasor reference. Let

$$\phi(t) = \phi_0 e^{-i\omega t}$$
$$V_m(t) = V_0 e^{-i\omega t}$$

then

$$\frac{V_0}{\phi_0} = \frac{1 - e^{i\omega\tau}}{1 - r_1 r_2 e^{i\omega\tau}} \qquad 1$$

if ω coincides nearly with a cavity free spectral range, and if $r_1 r_2 \simeq 1$, this transfer function can be approximated by

$$\frac{V_0}{\phi_0} \simeq \frac{-i\delta\tau_s}{1 - i\delta\tau_s} \qquad 2$$

where $\delta \equiv \omega - \frac{2\pi n}{\tau}$ and $\tau_s \equiv \frac{\tau}{1 - r_1 r_2}$.

2 Servo Stability

In this section we explore the stability of a servo loop containing the transfer function (1). We assume a cavity 4 km long, with $r_1 r_2 = 0.95$. Bode and Nyquist plots for $\frac{V_0}{\phi_0}$ are shown in Fig. 1 (that the Nyquist plot is exactly a circle can be seen from the fact that (1) is a Möbius transform in the variable $e^{i\omega\tau}$, which, as a function of ω, maps the real line into a circle in the complex plane). The open loop gain, H, of the servo-locked system will be $H(\omega) = \frac{V_0(\omega)}{\phi_0(\omega)} G(\omega)$, where $G(\omega)$ is the gain of the feedback amplifier. A simple feedback amplifier might have $G(\omega) = \frac{K}{-i\omega\tau_s}$; this gives

$$H(\omega) = \frac{K}{-i\omega\tau_s} \frac{1 - e^{i\omega\tau}}{1 - r_1 r_2 e^{i\omega\tau}} \quad .$$

Bode and Nyquist plots for $H(\omega)$ with $K = 200$ are shown in Fig. 2. Clearly the phase margin becomes small near the resonance at the first free spectral range. One solution to this problem is to add pole-zero pairs in order to reduce the phase lag at frequencies above the first resonance. Bode and Nyquist plots for

$$H(\omega) = K \frac{1 + \frac{i\omega}{z_1}}{1 + \frac{i\omega}{p_1}} \frac{1 + \frac{i\omega}{z_2}}{1 + \frac{i\omega}{p_2}} \frac{1}{-i\omega\tau_s} \frac{1 - e^{i\omega\tau}}{1 - r_1 r_2 e^{i\omega\tau}}$$

with $z_1 = (2\pi)\,40\text{kHz}$, $p_1 = (2\pi)\,400\text{kHz}$, $z_2 = (2\pi)\,4\text{MHz}$, $p_2 = (2\pi)\,40\text{MHz}$, and $K = 200$, are shown in Fig. 3 (the curves are traced only up to a frequency of 750 kHz because of the computational expense of tracing the rapidly increasing number of resonances in each octave of increasing frequency). The drawback to this feedback amplifier frequency response is that the slope of its gain above the first resonance is constrained to approximately -10dB per decade and one needs to increase the unity gain frequency considerably for a small increase in gain at the first resonance. The unity gain frequency will probably be limited to a few MHz by technical constraints, and the gain achievable at the first resonance will not be large except for very long cavities. In principle one could also increase the bandwidth past the first resonance by inserting poles and zeroes into the feedback amplifier frequency response to cancel the poles and zeroes given by equation (2). Whether one can generate poles and zeroes with sufficiently high Q's and keep them tuned is uncertain.

3 Acknowledgements

I would like to thank Alex Abramovici, Lisa Sievers, and Mike Zucker for helpful discussions.