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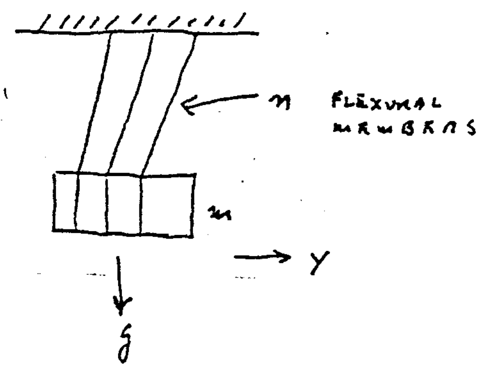
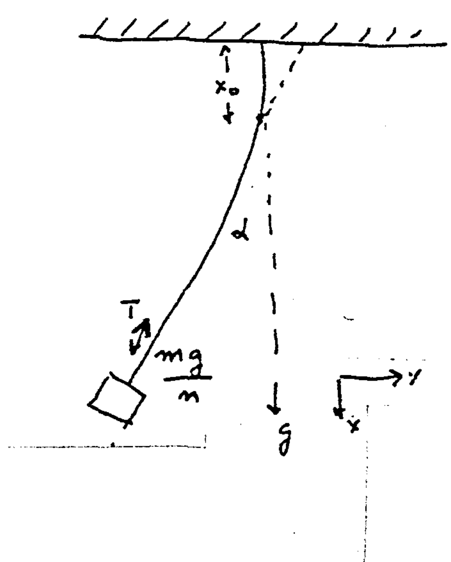
List of 5/01/89

Thermal noise in fiber flexure pendulum suspensions



Flexure members either circular or twisted ribbons

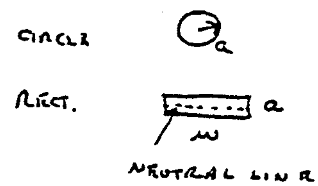
Equation for single member



$$y(x) = \alpha(x - x_o) + \alpha x_o e^{-x/x_o}$$

x_o is the shortening length of the pendulum due to the flexure

$$x_o = \left(\frac{yA\kappa^2}{mg/n}\right)^{1/2}$$



Y = Young's modulus
A = cross section of member
κ = radius of gyration of flexural member

$A = \pi a^2$ $\qquad \kappa = \frac{a}{2}$

$A = \omega a$ $\qquad \kappa = \frac{a}{\sqrt{12}}$ $\qquad \kappa^2 = \frac{1}{A}\int z^2 dA$

Energy stored in the flexure

$$E_{\text{elastic/member}} = \frac{1}{2}AY\kappa^2 \int_o^{\ell} \frac{\partial^2 y}{\partial x^2} dx = \frac{1}{2}AY\kappa^2 \int_o^{\ell} \left(\frac{\alpha}{x_o}\, e^{-x/x_o}\right)^2 dx$$

$$\cong \frac{1}{4}\frac{YA\kappa^2\alpha^2}{x_o} = \frac{1}{4}\left(\frac{mg}{n}YA\kappa^2\right)^{1/2}\alpha^2$$

Total elastic energy stored

$$E_{\text{elastic}} = n\,E_{\text{elastic/member}} = \frac{1}{4}(nmgYA\kappa^2)^{1/2}\ \alpha^2$$

Limiting dimensions for members is break stress

S_m = maximum stress in member

Condition for member geometry

$$\frac{mg}{n} = S_m A$$

Specify member geometry

<u>circular fiber</u>

$$a = \left(\frac{mg}{nS_m\pi}\right)^{1/2}$$

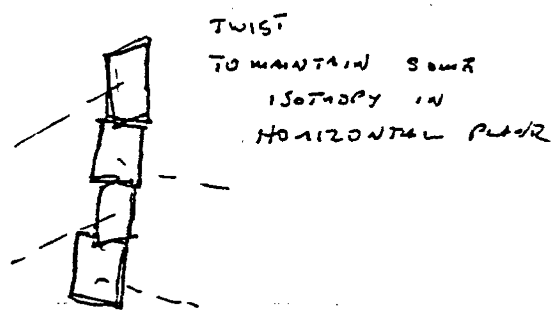
$$E_{\substack{\text{elastic}\\ \text{circular}}} = \left(\frac{Y}{n\pi}\right)^{1/2} \quad \frac{(mg)^{2/3}}{8S_m}\alpha^2$$

<u>Twisted rectangular fiber ribbon</u>

$$\frac{mg}{n} = S_m w a$$

$$a = \frac{mg}{nS_m w}$$

$$E_{\substack{\text{elastic}\\ \text{rectangular}}} = \frac{1}{4}\left(\frac{Y}{12}\right)^{1/2} \quad \frac{(mg)^{\cdot 2}}{nw(S_m)^{2/3}}\alpha^2$$



Ratio of elastic energy for the two geometries

$$\frac{E_{\text{rectangular}}}{E_{\text{circular}}} = 2\left(\frac{\pi}{12}\right)^{1/2} \left(\frac{mg}{nS_m}\right)^{1/2} \frac{1}{w}$$

Gravitational energy stored in the pendulum

$$E_{\text{grav}} = \frac{1}{2} m \; g \; \ell \; \alpha^2$$

Power lost in flexure in each member when oscillating

$$F_{\text{friction}} = -\beta v$$

$$P = F \cdot v = \langle \beta v^2 \rangle = \tfrac{1}{2} \beta \omega_o^2 \ell^2 \alpha^2$$

average over 1 period

The internal Q of the flexual member defines the power loss

$$P = \frac{\omega_o E_{\text{elastic}}}{Q_{\text{mat}}} \qquad Q_{\text{mat}} = \text{internal Q of material}$$

$$\beta = \frac{2E_{\text{elastic}}}{Q_{\text{mat}} \omega_o \ell^2 \alpha^2} \qquad \text{(force/velocity)}$$

The explicit values for

$$\beta_{\text{circular}} = \frac{1}{4} \left(\frac{Y}{n\pi}\right)^{1/2} \frac{(mg)^{3/2}}{S_m \omega_o \ell^2 Q_{\text{mat}}}$$

$$\beta_{\text{rectangular}} = \frac{1}{2} \left(\frac{y}{12}\right)^{1/2} \frac{(mg)^2}{nw(S_m)^{3/2} \omega_o \ell^2 Q_{\text{mat}}}$$

Thermal noise from the members added incoherently

$$F^2(f) \Big/ member = 4kT\beta$$

Total thermal noise driving term (incoherent sum)

$$F^2(f) = nF^2(f) \Big/ member = 4kT\beta n$$

Thermal displacement noise

$$x^2(f) = \frac{F^2(f)}{m^2\left[(\omega_o^2-\omega^2)^2+\left(\frac{\beta n}{m}\omega\right)^2\right]}$$

↑damping added coherently

Off resonance case $\omega \gg \omega_o$ $\quad x^2(f) = \frac{F^2(f)}{m^2\omega^4}$

<u>circular</u>

$$x^2(f) = kT\left(\frac{nY}{\pi}\right)^{1/2} \frac{\omega_o^3}{(mg)^{1/2}S_m Q_{\text{mat}}} \quad \frac{1}{\omega^4} \qquad x(f) \propto \frac{n^{1/4}}{m^{1/4}}$$

<u>rectangular</u>

$$x^2(f) = 2kT\left(\frac{Y}{12}\right)^{1/2} \frac{\omega_o^3}{S_m^{3/2} w Q_{\text{mat}}} \quad \frac{1}{\omega^4} \qquad x(f) \text{independent of } m, n$$

Possible examples

$$
\begin{aligned}
Q_{\text{mat}} &= 10^4 \\
m &= 10kg = 1 \times 10^4 gm \\
S_m &= 1 \times 10^{10}\ dynes/cm^2 \\
Y &= 2 \times 10^{12}\ dynes/cm^2 \\
w &= 1mm \\
n_{\text{rect}} &= 4 \\
T &= 300k \\
\omega &= 2\pi
\end{aligned}
$$

$$x(f) = 5.11 \times 10^{-12}/\omega^2$$

$$\underset{\text{circular}}{x^2(f)} = \frac{2.61 \times 10^{-23}}{\omega^4}$$

$$
\begin{aligned}
x(10) &= 1.3 \times 10^{-15} cm/H_2^{1/2} \\
x(100) &= 1.3 \times 10^{-17} cm/H_2^{1/2} \\
x(1kH_2) &= 1.3 \times 19^{-19} cm/H_2^{1/2}
\end{aligned}
$$

$$x(f) = 2.9 \times 10^{12}/\omega^2$$

$$\underset{\text{rectangular}}{x^2(f)} = \frac{8.37 \times 10^{-24}}{\omega^4}$$

$$
\begin{aligned}
x(10) &= 7.3 \times 10^{-16} cm/H_2^{1/2} \\
x(100) &= 7.3 \times 10^{-18} \\
x(1kH_2) &= 7.3 \times 10^{-20}
\end{aligned}
$$

What is the Q of the pendula with these parameters

$$Q \cong Q_{\text{mat}} \frac{E_{\text{gravity}}}{E_{\text{elastic}}} = \begin{cases} \text{circular} & \dfrac{Q_{\text{mat}} 4 S_m g}{\left(\frac{Y}{n\pi}\right)^{1/2} (mg)^{1/2} \omega_o^2} \\ \text{rectangular} & \dfrac{Q_{\text{mat}} 2 S_m^{3/2} wng}{\left(\frac{Y}{n\pi}\right)^{1/2} mg\omega_o^2} \end{cases}$$

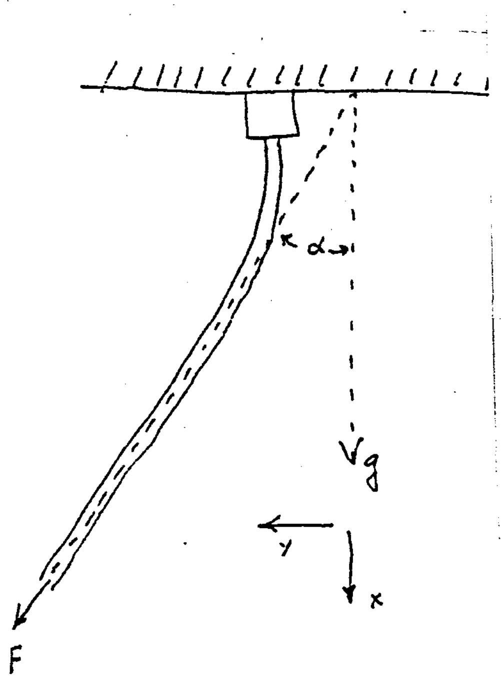
Field relieved mass?
to make g effective
smaller

Example

$$Q_{\text{circular}} \simeq 8 \times 10^6 \qquad Q_{\text{rectangular}} = 5.1 \times 10^7$$

ELASTICITY OF A FIBER OR RIBBON

NOTE: Thermal noise at lower flexure not included in analysis.

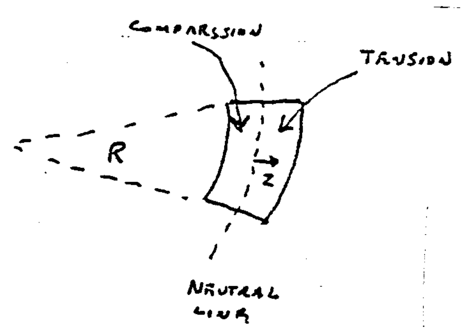


Equation for Shape of Fiber

$$y = f(x)$$

Asymptotic Limit $y = \alpha x$

External bending moment at any point x along the fiber due to force F which is along fiber.



$$M_{ext} = F(y - \alpha x)$$

The internal bending moment due to deformation of the fiber about the neutral line.

$$M_{\text{int}} = \frac{Y\ A\ \kappa^2}{R} = Y\ A\ \kappa^2 \ \frac{\partial^2 y}{\partial x^2}$$

κ is the radius of gyration defined by

$$\kappa^2 = \frac{1}{A} \int z^2 \, dA$$

At equilibrium moments balance at all points $M_{ext} = M_{int}$

$$\frac{\partial^2 y}{dx^2} = \frac{F}{Y\ A\ \kappa^2} (y - \alpha x)$$

Y = Young's Modulus
A = cross section $\perp$ to neutral line

Note: Fiber in pendulum is always in equilibrium. Dynamics becomes important when near first string mode.

Define

$x_o = \left(\frac{Y\ A\ \kappa^2}{F}\right)^{\frac{1}{2}}$

Boundary conditions:

$$x = 0, \quad \frac{dy}{dx} = 0$$
$$x \gg x_o \quad y = \alpha x$$

Solution to differential equation that satisfies boundary condition

$$y = \alpha x + \alpha x_o\, e^{-x/x_o}$$

Sample values of x_o: Tungsten at yield stress

Assume yield stress

$$S = 2 \times 10^{10}\ dynes/cm^2$$
$$Y = 2 \times 10^{12}\ dynes/cm^2$$

n = 4 fibers with 10kg mass

Radius of Circular Fiber

$a = \left(\frac{mg}{m\ S_m \pi}\right)^{1/2} = 6.2 \times 10^{-3} cm \qquad F = \frac{mg}{4}$

$\kappa^2 = a^2/4 = 9.75 \times 10^{-6} cm^2$

$A = \pi a^2 = 1.23 \times 10^{-4} cm^2$

$x_o = 3.13 \times 10^{-2} cm$