LIGO-T880005

03

List of 3/1/89

A Model Calculation of Vibration Isolation from Bellows

§1 Summary

I give a method for calculating the vibration isolation between the vacuum pipes and the instrumentation tanks provided by compliant bellows and a massive stiff anchor adjacent to the tank. The importance of the existence of internal resonances in the various components is made clear. Nevertheless, good order-of-magnitude estimates can be made using only a small number of parameters to characterize the components.

§2 Introduction

At various stages in our thinking about the LIGO project, questions have been raised about what effect mechanical noise in the vacuum pipes may have on the stochastic force noise spectrum of the test masses. We have always imagined that there would be compliant bellows in the pipe to take up low frequency length changes due to temperature excursions. Opinions of the effectiveness of these bellows for vibration isolation have spanned the range from 'worthless' to 'panacea'. The purpose of this memo is to show by calculation what range of performance we are actually likely to encounter.

§3 Lumped-Parameter Model

I imagine that the tank will be connected to a pair of compliant bellows. Between the two bellows is a short section of pipe which is connected to the ground by a massive, stiff anchor. A sketch of the scheme is shown in Figure 1.

I start with a simple model consisting only of masses connected to massless springs. Figure 2 shows the model I chose. The bellows themselves are modelled as ideal springs. The anchor and the tank are each treated as a mass attached to the ground through an ideal spring. I take the motion of the 'outboard' end of the first bellows as the input, and the motion of the tank mass as the output.

For simplicity I will treat the motion of each mass as one-dimensional. With appropriate choices of spring constants, this could represent either transverse or longitudinal motion.

Now it is a straightforward matter to write down Newton's Second Law for the two masses in the problem. It gives

$$(-M_a\omega^2 x_a + k_a x_a) + k_b(x_a - x_i) + k_b(x_a - x_t) = 0 \tag{1}$$

and

$$(-M_t\omega^2 x_t + k_t x_t) + k_b(x_t - x_a) = 0, \tag{2}$$

where I have assumed a steady input at frequency ω so that I could replace accelerations by $-\omega^2 x$.

Solving these coupled linear equations for the transfer function x_t/x_i (using Cramer's rule, for example) gives

$$\frac{x_t}{x_i} = \frac{\omega_{ba}^2\omega_{bt}^2}{\omega^4 - \omega^2(\omega_{aa}^2 + \omega_{tt}^2 + \omega_{bt}^2 + 2\omega_{ba}^2) + \omega_{ba}^2\omega_{bt}^2 + \omega_{aa}^2\omega_{bt}^2 + \omega_{aa}^2\omega_{tt}^2 + 2\omega_{ba}^2\omega_{tt}^2}. \tag{3}$$

(Here $\omega_{ij}^2 = k_i/M_j$, where i can refer to the bellows, the anchor, or the tank, and j can specify either the anchor or the tank.) The transfer function contains two resonances (of infinite Q in this example since we haven't included any damping terms in the equations). Below the resonances, there is a constant transmission equal to $k_b^2/k_a k_t$. Above the resonances, the transmission falls as ω^{-4}. Since the bellows will have a much lower spring constant than the anchor or the tank, the transmission starts out low at low frequencies and gets lower at high frequencies (except at the resonances). A graph of this transfer function is shown in Figure 3. The values for the physical parameters were $k_b = 3.77 \times 10^8$, $k_a = 1.25 \times 10^{11}$, and $k_t = 2.45 \times 10^{11}$, all in dynes/cm, and $M_a = M_t = 2$ tonnes. (The number of significant figures is misleading; I just made up what seemed like reasonable round numbers for resonant frequencies. Note that I took the tank to be rather stiffly connected to the ground. It would be easy to substitute numbers for a tank on an air spring system, for example.)

§4 Distributed-Parameter Model

The lumped component model ignores the internal modes of vibration of the components. The degree to which the internal modes compromise the ideal performance calculated above is the technical question whose answer determines the severity of the transmitted vibration problem. An exact calculation using the normal modes of the actual structures would be very difficult. Fortunately it is not necessary. All of the important features are exhibited in a model which is very simple. The key to this calculation is to adopt the concept of mechanical impedance, and to use a simple model for the impedance of a distributed parameter system (i.e. a spring with finite mass or a mass with finite length and compliance).

First, I recast the problem solved in the previous section in the language of mechanical impedance. Impedance is defined as the ratio of the driving force on a system to the resulting velocity of the system. Impedances are further characterized as driving point impedances, where the force and velocity are measured at the same point, or as transfer impedances, where the two quantities are measured at different points. Hooke's Law can be expressed by saying an ideal spring has a driving point impedance of $Z_k = -k/i\omega$, with a transfer impedance of the opposite sign. A damper has a real, frequency independent impedance, again negative for the driving point impedance and positive for the transfer impedance. Newton's Second Law says an ideal mass has impedance $Z_M = i\omega M$. (An ideal mass has only a driving point impedance since it is by definition pointlike or at least rigid.)

Using the definition of impedance given above, we could rederive the results of the previous section. Figure 4 shows an appropriately abstract system model. Newton's Law looks like

$$\sum_i Z_i v_i = 0 \tag{4}$$

where there is one term for each driving point impedance connected to the point in question, and one term for each transfer impedance. (One of the terms at each point will be $i\omega M_i v_i = Ma_i$.) If we carried out this procedure, writing one equation for the nominal anchor location and one for the tank, we would find the same transfer function as before.

The point of this exercise, though, is to add some extra physical content. What this method allows us to do is to include, in a tractable form, a model for the internal resonances of real masses and springs. A simple expression is derived by Molloy (1957, *J. Acoust. Soc. Am.* vol.29, p.842). He shows that for a one-dimensional object with distributed compliance and mass one can write the force at one end as

$$F_2 = \left(\frac{-i\sqrt{km}}{\sin\frac{\omega}{\omega_r}}\right)v_1 + \left(i\sqrt{km}\cot\frac{\omega}{\omega_r}\right)v_2. \tag{5}$$

The coefficient of v_1 is the transfer impedance from end 1 to end 2, and the term multiplying v_2 is the driving point impedance. These impedances exhibit resonances which are evenly spaced harmonics of $\omega_r = \sqrt{\frac{k}{m}}$, as is appropriate for the simple one dimensional model. This will have sufficient richness for our needs.

The impedance form of Newton's Second Law, applied to the configuration of Figure 4, becomes

$$Z_a^p v_a + 2Z_b^p v_a + Z_b^t v_i + Z_v^t v_t = 0 \tag{6}$$

and

$$Z_t^p v_t + Z_b^p v_t + Z_b^t v_a = 0, \tag{7}$$

where the superscripts p and t refer to driving point and transfer impedances, respectively, and the subcripts i, b, a, and t refer to input, bellows, anchor, and tank. The transfer function can be found using the same method as in the lumped-parameter case. Here, it is

$$\frac{x_t}{x_i} = \frac{v_t}{v_i} = \frac{Z_b^{t2}}{Z_a^p Z_t^p + Z_a^p Z_b^p + 2Z_t^p Z_b^p + 2Z_b^{p2} - Z_b^{t2}}. \tag{8}$$

The transfer function has the same low frequency limit as in the lumped-parameter model. The high frequency structure is quite different. In this more realistic model, the transmission above the lowest resonance fluctuates strongly. A graph of this function is shown in Figure 5, for the same sample parameters used in the lumped-parameter model. Rough order-of-magnitude reasoning would suggest that the average transmission in the high frequency region should be equal to $k_b M_b/\sqrt{k_a M_a}\sqrt{k_t M_t}$ (by analogy with the exact result for the low frequency region $k_b^2/k_a k_t$). The calculation shows that the typical high frequency transmission is somewhat lower than this argument predicts. Nevertheless, it is clear that the transmission is much higher in this more realistic model than in the calculation where resonances are ignored.

So far no damping has been included in the model. It can be included in various ways. One simple model replaces the real spring constant k by a constant with both real and imaginary parts, which can be written as $k(1+i\gamma)$. A problem with this is that the Q of the modes becomes a strong function of frequency. An expression which doesn't suffer from this problem is $k(1+\frac{i\gamma}{1+\frac{\omega}{\omega_r}})$. Figure 6 shows a transfer function calculated using this model, for the rather heavily damped case with γ of 0.2 for the bellows and tank, and 1 for the anchor. (These numbers correspond to Q of about 10 in the bellows and tank, and even smaller in the anchor.)

§5 Cautions and Discussion

I think that the basic method proposed here is sound, if the goal is just to obtain approximate numerical estimates for the vibration transmission through bellows. It is important to see what the expressions give for the parameters of actual hardware.

The inclusion of damping was done in a rather *ad hoc* fashion. For this reason especially (and from a distrust of simple models generally), it would be nice to test the model with an experiment. Probably an experiment with small components would be sufficient to test the model.

Peter R. Saulson, 21 June 1988, corrected 2 Feb 1989

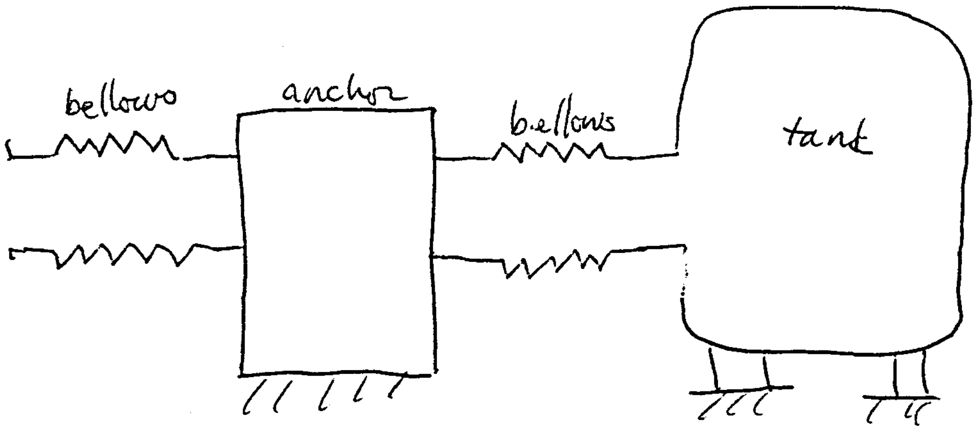

Figure 1: Model bellows configuration

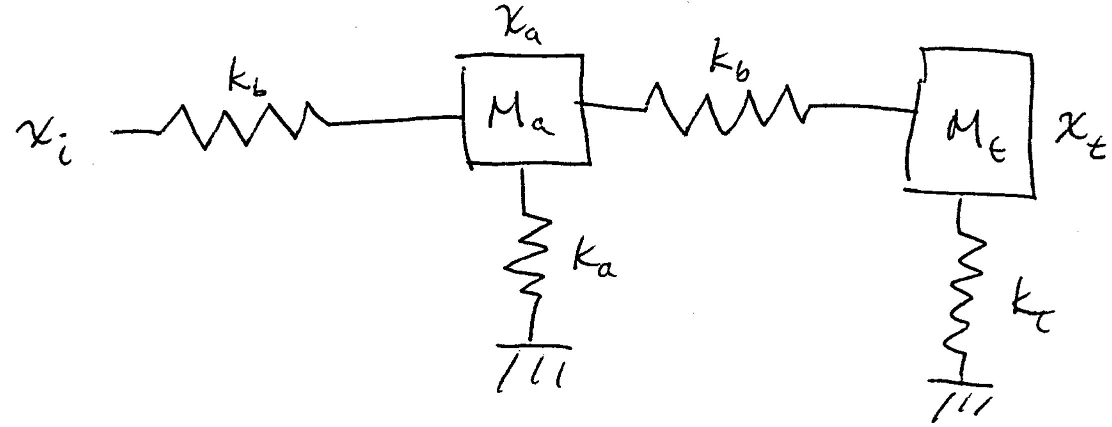

Figure 2: Lumped-Parameter Model

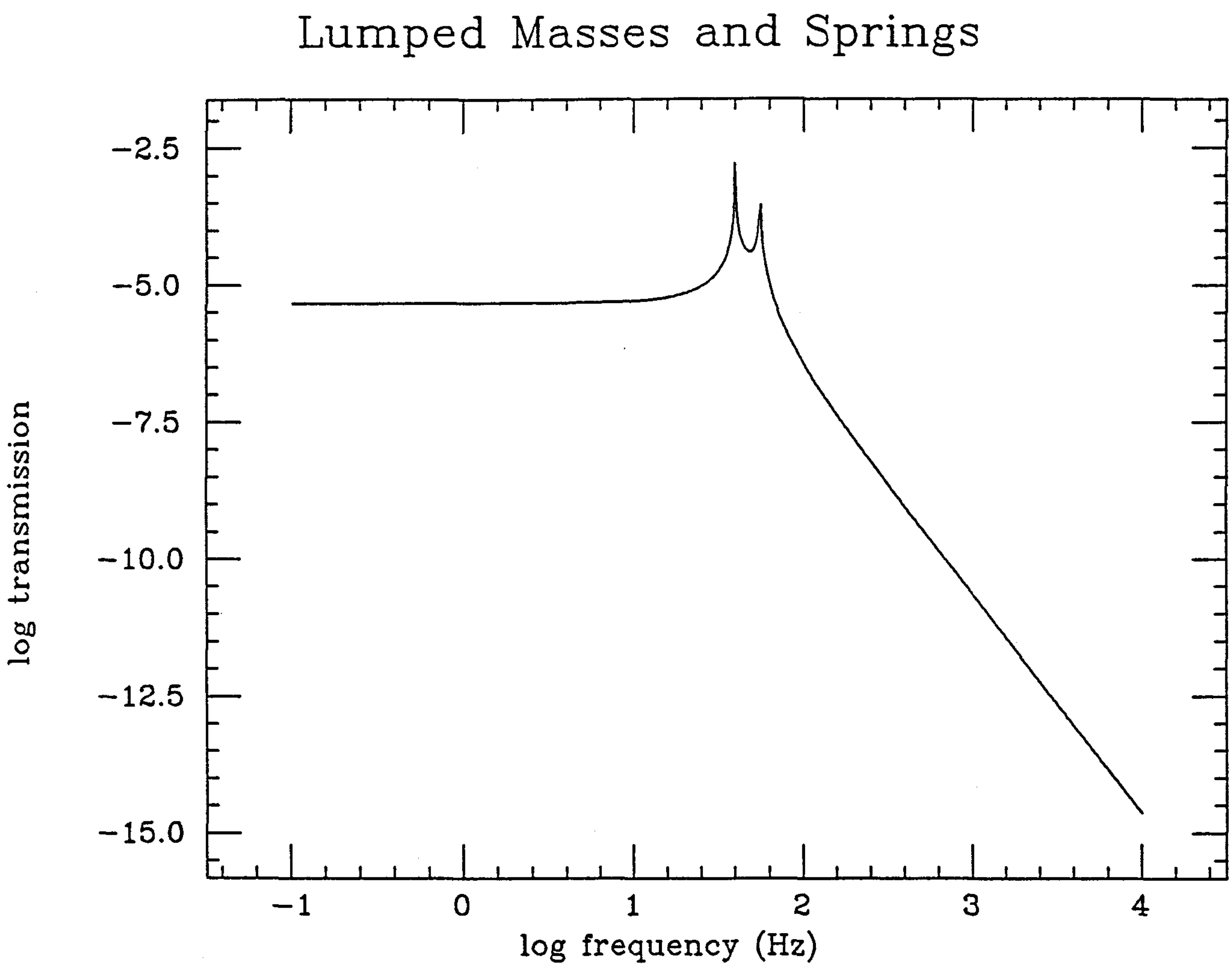

Figure 3: Transfer Function for Lumped-Parameter Model

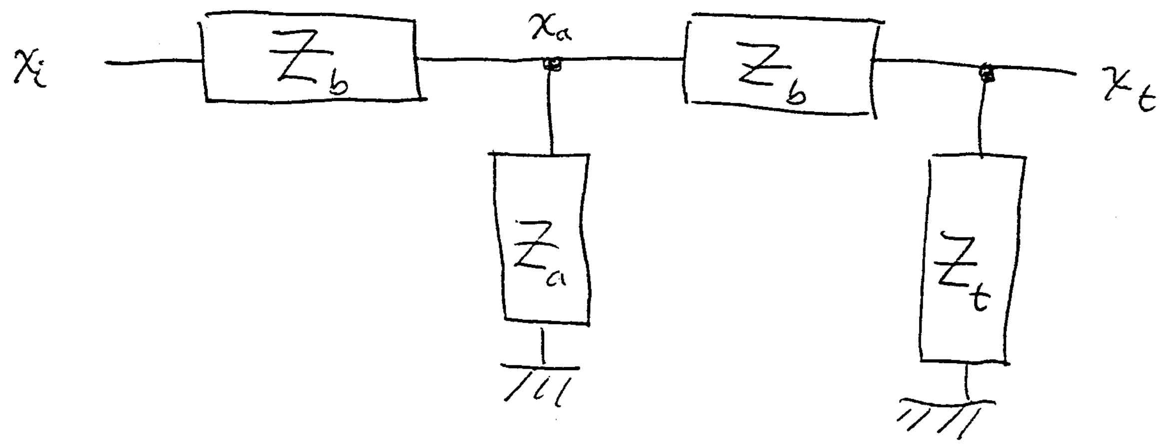

Figure 4: Impedance Model

Internal Modes Modelled in All Components

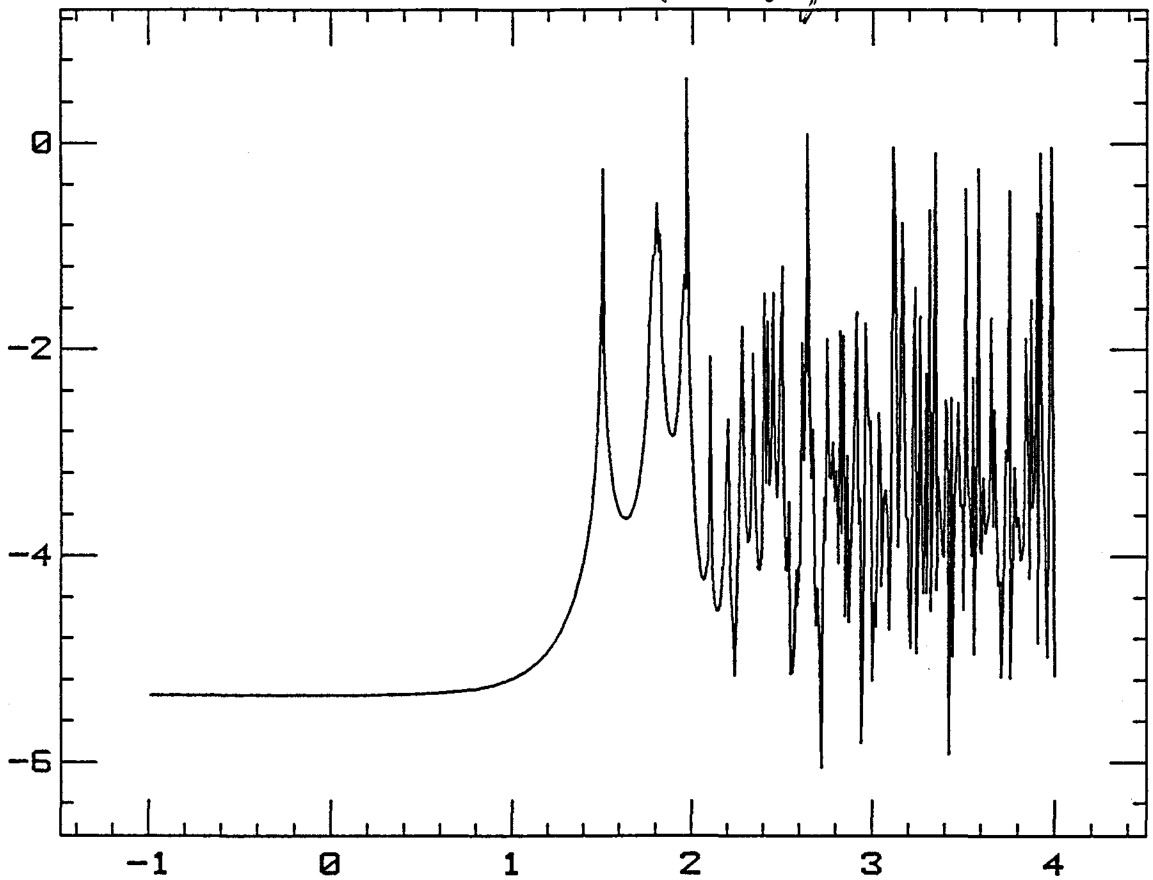

Figure 5: Transfer Function for Distributed Parameter Model
No damping

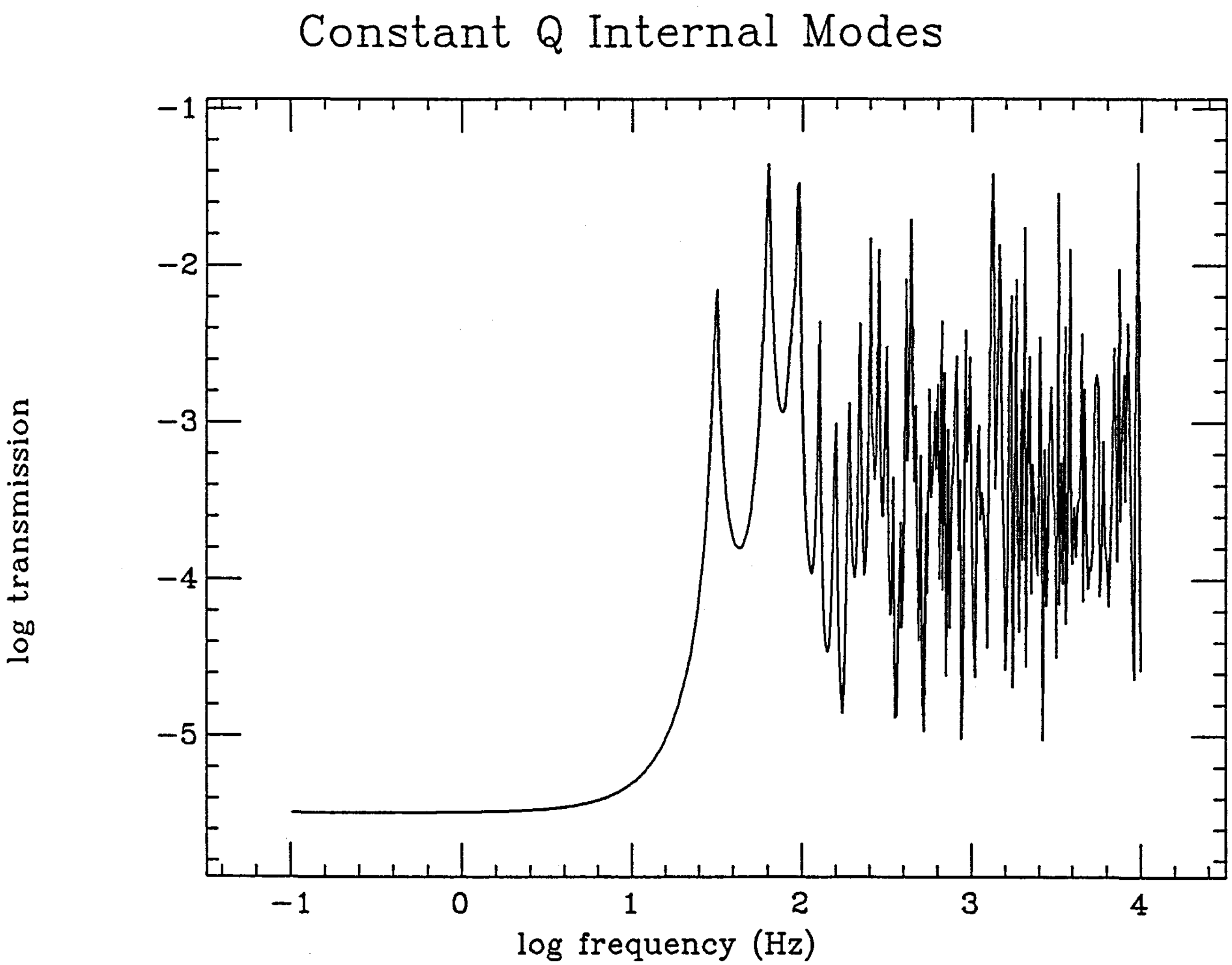

Figure 6: Transfer Function for Distributed Parameter Model
Strong Damping