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Shot Noise in the Caltech Gravitational Wave Detector—The mid-1984 Configuration

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Abstract

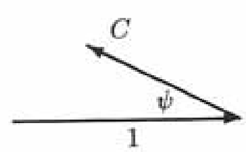
The goal of this note is to calculate the shot noise for the Caltech interferometer in the configuration it is actually used now.

Introduction

The present configuration of the Caltech interferometric gravitational wave detector is shown in Figure 1. Two 40 m long optical cavities define the arms of the detector. Separate photodiodes monitor the reflected light from the first mirrors of each cavity. The first arm is used to stabilize the laser frequency while the second arm measures the difference between the laser frequency and the cavity resonance. The comparisons are made using a phase modulation technique. The goal of this paper is to calculate the shot noise for the current interferometer configuration, including the effects of the modulation scheme and certain imperfections in the interferometer (most notably, the nonideal fringe visibility).

The Modulation Scheme

Consider arm 1 first. The input laser light is phase modulated at ~ 12 MHz, a frequency high compared with the cavity bandwidth.



Assume that the incident laser light has α_1 photons/sec. The beam incident on the cavity can be decomposed into two kinds of light—a fraction β^2 which is properly mode-matched to the cavity and a fraction $1-\beta^2$ which is not. The incorrectly matched part will be entirely reflected onto the photodiode.

The properly matched light falling on the photodetector is due to the interference between two amplitudes—the reflected laser light with its phase-modulation and the cavity light which has no phase modulation. The intensity on the photodiode is just the sum of the intensities of the properly mode-matched light and the improperly mode-matched light; there is no interference between these two pieces of the light because they are in spatial orthogonal modes (provided the photodiode is large enough to intercept the full beam). Thus the expected intensity on the photodiode is

Cavity Amplitude $= C$
Laser Amplitude $= 1$
Sum (matched):
$\beta^2(1 + C^2 - 2C\cos\psi)$
where
$C = AJ_0(\Phi_m)$
$\psi = \Phi_m \sin\omega_m t$

Mismatched:
$1-\beta^2$

$$\langle\eta_1(t)\rangle = \alpha_1\left[1 + \beta^2 A^2 J_0^2(\Phi_m) - 2\beta^2 A J_0(\Phi_m)\cos(\Phi_m \sin\omega_m t)\right] \tag{1}$$

where A is the amplitude the cavity light would have in resonance if there were no modulation. J_0 is a Bessel function of zero order, and Φ_m and ω_m are the amplitude and frequency of the phase modulation. A is determined by the properties of the cavity mirrors,

$$A = \frac{|t_1|^2 |r_2|}{1-|r_1 r_2|} \tag{2}$$

here $t_1, r_1,$ andr_2 are the amplitude transmissivity and reflectivity of the cavity mirrors. The amplitude of the cavity light is reduced by a factor of $J_0(\Phi_m)$, the fraction of the incident field which is not shifted to one of the sidebands by the phase modulation. The actual number of photons falling on the detector $\eta_1(t)$ will vary about the expected number. The rms variation of η_1 in a short interval dt will have the value given by Poisson statistics, namely $[\eta_1(t)dt]^{1/2}$. the amplitude the cavity light would have in resonance if there were no modulation. J_0 is a Bessel function of zero order, and Φ_m and ω_m are the amplitude and frequency of the phase modulation.

Internal Field:

$$t_1\left(1 + r_2 + r_2 r_1 r_2 + r_2 (r_1 r_2)^2 + \ldots\right) \approx t_1\left(1 + r_2\left[\frac{1}{1-r_1 r_2}\right]\right) \approx \frac{t_1 r_2}{1-r_1 r_2}$$

$A = t_1 \times$ Internal Field

$L + R + T = 1$
$T_2 = 0,\ L_1 = L_2 = L$
$\Rightarrow A = \left(\frac{1}{2} + \frac{L}{T_1}\right)^{-1}$

$$0 < A < 2$$

The effect of a gravity wave is to cause a phase difference ϕ_1 between the laser light and the cavity light (Figure 2b). In general, the phase difference in arm 2 caused by the gravity wave will have the opposite sign from ϕ_1, and it is the difference $\phi_1 - \phi_2$ which we measure. This phase difference can be related to the gravity wave amplitude $h(\omega)$ by a transfer function[1] which depends on

i.e. $\psi \rightarrow \psi + \phi_1$

[1]The transfer function shown assumes that $h(t)$ and $\Delta\phi(t)$ are related to $h(\omega)$ and $\Delta\phi(\omega)$ by

$$f(t) = \int f(\omega) e^{-i\omega t} d\omega.$$

Engineers may prefer a plus sign in the exponential, in which case they should change the frequency and angle of incidence of the wave (preprint). The most favorable case is that of a wave incident normal to the plane of the detector. With the additional assumption that the detector is small compared to the gravitational wavelength, the phase differences are given by $\phi_1 = -\phi_2$ and

$$\begin{aligned}\Delta\phi(\omega) &= \frac{4\pi L}{\lambda}\frac{e^{i\omega\tau_t}\left(1-|r_1 r_2|\right)}{1-2|r_1 r_2|e^{i\omega\tau_t}\cos\omega\tau_t + |r_1 r_2|^2 e^{i2\omega\tau_t}}h(\omega) \\ &\approx \frac{4\pi L}{\lambda}\left[\frac{\tau_t}{\tau_s} - 2i\omega\tau_t\right]^{-1} h(\omega) \qquad (3)\end{aligned}$$

where L is the length of one arm, $\tau_t = L/c, \text{and} \tau_s$ is the cavity storage time,

$$\tau_s = \frac{\tau_t}{1-|r_1 r_2|} \qquad (4)$$

The noise level is frequently expressed as a displacement (m $\mathrm{Hz}^{-1/2}$) using the relation

$$\delta l = hL. \qquad (5)$$

The photon flux incident on photodetector 1 in the presence of a wave is

$$\langle\eta_1(t)\rangle = \alpha_1\left[1+\beta^2 A^2 J_0^2 - 2\beta^2 A J_0 \cos\left(\phi_1(t) + \Phi_m \sin\omega_m t\right)\right] \qquad (6)$$

where all bessel functions are henceforth understood to be evaluated at Φ_m. For any realistic case, $\phi_{1,2} \ll 1$, and we can expand to first order in $\phi_{1,2}$

$$\begin{aligned}\langle\eta_1(t)\rangle \approx\ & \alpha_1\Big[1+\beta^2 A^2 J_0^2 - 2\beta^2 A J_0 \cos\left(\Phi_m \sin\omega_m t\right) \\ & + 2\phi_1(t)\beta^2 A J_0 \sin\left(\Phi_m \sin\omega_m t\right)\Big] \qquad (7)\end{aligned}$$

The signal from the photodiode is demodulated using a mixer. The process is equivalent to multiplying by $\sin\omega_m t$ and averaging with t time constant long compared to ω_m^{-1} but short compared to the time scales interesting for gravitational wave detection.

n'th bounce amplitude

$$\begin{aligned}a(n) &= (r_1 r_2)^n \\ &\approx e^{-n(1-r_1 r_2)} \\ I(t) &= a^2\left(n = \frac{t}{2\tau_t}\right) \\ &= e^{-\frac{t}{\tau_t}(1-r_1 r_2)}\end{aligned}$$

$\tau_s =$ *energy* storage time.
$A = T_1 \frac{\tau_s}{\tau_t}$

$\cos(x+\epsilon) = \cos x - \epsilon \sin x$

Signal

Assume a sinusoidal signal

$$\phi_1 = -\phi_2 = \Delta\phi_0 \sin\omega t \qquad (8)$$

ω to $-\omega$ in the transfer function.

$\Delta\phi_0$ can be measured by comparing signals S_1 and S_2 derived from the two photodetectors,

$$S_1(\omega) = \frac{1}{T}\int_0^T \eta_1(t)\sin\omega_m t \sin\omega t\, dt = \frac{\alpha_1\beta^2 A J_0 J_1 \Delta\phi_0}{2} \qquad (9)$$

It follows trivially that

$$\Delta\phi_0 = \frac{S_1(\omega)}{\alpha_1\beta^2 A J_0 J_1} - \frac{S_2(\omega)}{\alpha_2\beta^2 A J_0 J_1} \qquad (10)$$

$$J_1(x) = \frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin\theta\, d\theta$$

Noise

To calculate the noise, assume $\phi_1 = \phi_2 = 0$. In this case the noise is formally defined to be teh sum of the squares of $\Delta\phi$ measured with both $\sin\omega t$ and $\cos\omega t$ phases, i.e.

$$N_\phi^2(\omega) = \left\langle\left[\frac{1}{T}\int_0^T\left(\frac{\eta_1}{\alpha_1}-\frac{\eta_2}{\alpha_2}\right)\frac{\sin\omega_m t\sin\omega t}{\beta^2 A J_0 J_1}\,dt\right]\right\rangle^2 + \left\langle\left[\frac{1}{T}\int_0^T\left(\frac{\eta_1}{\alpha_1}-\frac{\eta_2}{\alpha_2}\right)\frac{\sin\omega_m t\cos\omega t}{\beta^2 A J_0 J_1}\,dt\right]\right\rangle^2 \qquad (11)$$

where <> denotes average value. (This is what we see on the spectrum analyzer.) To evaluate N_ϕ^2 we can use a generalization of an elementary formula for the propagation of errors. If

$$y = \sum \alpha_i x_i$$

then

$$\sigma_y^2 = \sum\left(\frac{\partial y}{\partial x_i}\right)^2 {\sigma_{x_i}}^2 = \sum \alpha_i^2 {\sigma_{x_i}}^2.$$

$$f(\Phi_m) = \frac{1 + J_0^2 - 2J_0^2 + 2J_0J_2}{2J_0^2J_1^2} = \frac{3}{2} + \frac{37\Phi_m^2}{48} + \frac{571\Phi_m^4}{2304} + \cdots$$

$f(1) = 2.603 \quad f(0.8) = 2.115$

Converting this to an integral, remembering that the uncertainty in $\eta(t)dt$ is $[\eta(t)dt]^{1/2}$, yields

$$N_\phi^2(\omega) = \frac{1}{T}\int_0^T\left(\frac{\eta_1}{\alpha_1^2}+\frac{\eta_2}{\alpha_2^2}\right)\frac{\sin^2\omega_m t}{\beta^4A^2J_0^2J_1^2}\,dt = \left[\frac{\beta^{-2} + A^2J_0^2 - 2AJ_0^2 + 2AJ_0J_2}{2\beta^2A^2J_0^2J_1^2}\right]\left[\frac{1}{\alpha_1}+\frac{1}{\alpha_2}\right] \qquad (12)$$

Notice that $N_\phi^2(\omega)$ has units of radians^2 Hz^{-1}.

The noise level can be converted to either displacement sensitivity

$$N_x^2(\omega) = \left[\frac{\lambda}{4\pi}\right]^2 \left[\frac{\tau_t^2}{\tau_s^2} + (2\omega\tau_t)^2\right] \times \left[\frac{\beta^{-2} + A^2J_0^2 - 2AJ_0^2 + 2AJ_0J_2}{2\beta^2A^2J_0^2J_1^2}\right]\left[\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right] \quad (13)$$

Perfect mirrors: $\tilde{h}(f) = \frac{1}{4\pi\tau_s}\left[\frac{3\lambda h}{2cP}\left(1 + 4^2\pi^2 f^2 \tau_s^2\right)\right]^{-1/2}$

or gravity wave sensitivity

$$N_h(\omega) = \frac{N_x(\omega)}{L}$$

1 Discussion

As an example, let's compare the theoretical shot noise with a measured value. The present cavities have a storage time of ~ 0.8 ms. The measured transmission of the input cavity mirror is $|t_1|^2 = 1.2 \times 10^{-4}$. these two values determine $A = 0.72$. The observed fringe visibility with no modulation ($= \beta^2[2A - A^2]$) of 0.5 determines $\beta^2 = 0.54$. The power in the interferometer is ~ 1 mw (corrected for photodiode efficiency) for one arm and ~ 0.5 mW for the other arm. The modulation index Φ_m is about 0.5. These values yield the curve shown in Figure 3. The curve for $\beta^2 = 1.0$ with all other parameters the same is also shown. This represents the expected improvement in the shot noise which can be achieved by improving the mode matching; it also represents teh best achievable sensitivity in this configuraiton with the present mirrors and laser power.

Visibility= $\beta^2 \frac{8T_1L}{(T_1+2L)^2}$

The measured curve in Figure 3 is found using a flashlight on the photodiode to give the same intensity on the photodiode as when the cavities are in resonance. (The difference between the flashlight (constant intensity) and the modulated intensity of the laser light is unimportant because of the poor fringe visibility.) The measured value agrees with the theoretical value within the uncertainties of the calibraiton and of the other parameters.

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References

Y. Gursel, P. Linsay, P. Saulson, R. Spero, R. Weiss, and S. Whitcomb, in preparation (1984).