LIGO-T080141-00-Z

Generalization of $\mathcal{F}$-Statistic and LALDemod

Iraj Gholami

May 14, 2008

Abstract

The $\mathcal{F}$-statistic, which was originally derived by Jaranowski, Królak and Schutz (JKS)[1], is the optimal statistic for the detection of nearly periodic gravitational waves from unknown GW pulsars. In fact, the $\mathcal{F}$-statistic derived by JKS applies just to the case of a single detector with stationary noise (*i.e.*, where spectral density is constant in time). Although JKS mentioned the possibility of using data from a network of detectors to calculate the $\mathcal{F}$-statistic, they did not give explicit details on how to implement this. The first detailed calculation for the general case of a network of detectors, with time-varying noise curves and, for a collection of known sources, was published by Cutler and Schutz (CS)[2]. In this document, we apply the formalism developed by CS to the case of a *single source* and N *detectors* with *uncorrelated noises*. The main aim of this document is to review the multi-IFO $\mathcal{F}$-statistic.

1 Maximum likelihood function ($\mathcal{F}$-statistic)

The basic problem in GW detection is to identify a gravitational waveform in a noisy background. Because all data streams contain random noise, the data are just a series of random values and therefore the detection of a signal is always a decision based on probabilities. The aim of detection theory is therefore to assess this probability.

The basic idea behind the current methods of signal detection is that the presence of a signal will change the statistical characterization of the data $x(t)$, in particular its probability distribution function (pdf) $P(x)$. Recall that the pdf is defined so that the probability of a random variable x_i lies in an interval between $x(t)$ and $x(t)+dx$ is $P(x)dx$. Let us denote by $P(x|0)$ the probability of a random process $x(t)$ (representing our data) in the absence of any signal, and by $P(x|h)$ the probability of that same process when a signal $h(t)$ is present. Given a particular measurement $x(t)$ obtained with our detector, is its probability distribution given by $P(x|0)$ or $P(x|h)$? In order to make that decision, we need to make a rule called a *statistical test*.

There are several approaches to finding an appropriate test, notably the Bayesian, Minimax and Neyman-Pearson approach (for an overview, we refer the reader to Jaranowski and Królak, [3] and the references listed therein). In the end, however, these three approaches lead to the same test, namely the *likelihood ratio test* [3, 4].

Among the three main approaches, the Neyman-Pearson approach is often used in the detection of gravitational waves [5]. This approach is based on maximizing the detection probability (equivalently minimizing the false dismissal rate) for fixed false alarm rate, where the detection probability is the probability that the random value of a process which contains the signal will pass our test, while the false alarm probability is the probability that data containing no signal will pass the test nonetheless. Mathematically, we can express these probabilities as [5]

$$P_D(R) = \int_R P(x|h)dx \qquad \text{Detection Probability,} \tag{1}$$

$$P_F(R) = \int_R P(x|0)dx \qquad \text{False Alarm Probability,} \tag{2}$$

where R is the detection region (to be determined).

The Likelihood ratio Λ is the ratio of the pdf when the signal is present to the pdf when it is absent:

$$\Lambda = \frac{P(x(t)|h(t))}{P(x(t)|0)}. \tag{3}$$

Writing the data as $x(t) = h(t) + n(t)$, with $h(t)$ represents the signal and $n(t)$ the noise, and with the assumption that the noise is a *zero-mean, stationary* and *Gaussian* random process, we can write the likelihood ratio as

$$\begin{aligned} \Lambda &= \frac{P(x|h)}{P(x|0)} \\ &= \frac{\exp(-\frac{1}{2}(x-h|x-h))}{\exp(-\frac{1}{2}(x|x))} \\ &= \exp[(x|h) - \frac{1}{2}(h|h)]. \end{aligned} \tag{4}$$

This leads to the log of likelihood function as

$$\log \Lambda = (x|h) - \frac{1}{2}(h|h). \tag{5}$$

The gravitational wave signal $h(t)$ depends nonlinearly on the amplitude parameters $(h_0, \psi, \iota, \Phi_0)$, but, crucially, one can make a simple change of variables – introducing new variables (A^1, A^2, A^3, A^4) – such that dependency of $h(t)$ is linear in the A^a [2]:

$$h(t) = \sum_{a=1}^{4} A^a h_a(t), \tag{6}$$

where the detector-dependent wave components $h_a(t) = h_a(\alpha, \delta, f, f^{(k)}; t)$ are given as

$$\begin{aligned} h_1(t) &= a(t)\cos\phi(t), &(7)\\ h_2(t) &= b(t)\cos\phi(t), &(8)\\ h_3(t) &= a(t)\sin\phi(t), &(9)\\ h_4(t) &= b(t)\sin\phi(t), &(10) \end{aligned}$$

with

$$\phi(t) = 2\pi \sum_{k=0}^{s} \frac{f^{(k)}(t^{(0)}_{\rm ssb})}{(k+1)!} \left(\Delta t_{\rm ssb}\right)^{k+1}, \tag{11}$$

and the constant (in time) amplitudes $A^a = A^a(h_0, \psi, i, \Phi_0)$ are [2]

$$\begin{aligned} A^1 &= A_+ \cos\Phi_0 \cos 2\psi - A_\times \sin\Phi_0 \sin 2\psi, &(12)\\ A^2 &= A_+ \cos\Phi_0 \sin 2\psi + A_\times \sin\Phi_0 \cos 2\psi, &(13)\\ A^3 &= -A_+ \sin\Phi_0 \cos 2\psi - A_\times \cos\Phi_0 \sin 2\psi, &(14)\\ A^4 &= -A_+ \sin\Phi_0 \sin 2\psi + A_\times \cos\Phi_0 \cos 2\psi. &(15) \end{aligned}$$

The quantities $a(t)$ and $b(t)$ are functions of right ascension α and declination δ and are independent of wave polarizations [1].

Using the above expressions, we can rewrite the simple expression of Eq. (5) for the likelihood function in terms of the new variables A^a we had introduced in Eq. (6). The result is

$$\log \Lambda = (x|A^a h_a) - \frac{1}{2}(A^a h_a|A^b h_b). \tag{16}$$

Since the A^as depend neither on the detector properties nor on the frequency or the time, we can take them out of the inner product and write the log of likelihood ratio as

$$\log \Lambda = A^a (x|h_a) - \frac{1}{2} A^a A^b (h_a|h_b). \tag{17}$$

Defining the new variables

$$H_a \equiv (x|h_a), \tag{18}$$

and

$$M_{ab} \equiv (h_a|h_b), \tag{19}$$

we have

$$\log \Lambda = A^a H_a - \frac{1}{2} A^a A^b M_{ab}. \tag{20}$$

The maximum detection probability follows from the maximization of the likelihood function: by maximizing the likelihood function with respect to the A^a (which, again, are independent of the detector), we have

$$\frac{\partial \log \Lambda}{\partial A^a} = 0. \tag{21}$$

This leads us to

$$H_a - A^b_{\mathrm{MLE}} M_{ab} = 0, \tag{22}$$

and therefore

$$A^b_{\mathrm{MLE}} = (M^{-1})^{ab} H_a. \tag{23}$$

The label *MLE* denotes the *Maximum Likelihood Estimator*; it corresponds to the values for the A^as we calculate from our data by maximizing the likelihood ratio (so that, in practice, we are calculating $A^a = E[A^a_{MLE}]$). By definition, the $\mathcal{F}$-Statistic is the *maximum of the logarithm of likelihood function.* Substituting Eq. (23) into Eq. (20), we have

$$\mathcal{F} \equiv \log \Lambda \mid_{\mathrm{MLE}} = \frac{1}{2}\ H_a\ (M^{-1})^{ab}\ H_b. \tag{24}$$

2 Generalized maximum likelihood function for multi-IFO

Going back to the definition of the maximum likelihood function in Eq. (3), and considering the fact that we are looking for a single signal buried in all the different data sets from the different detectors, the generalized likelihood function can be written as

$$\Lambda = \frac{P(x^1, x^2, \cdots |h)}{P(x^1, x^2, \cdots |0)}, \tag{25}$$

where x^1 stands for $x^1(t)$, x^2 for $x^2(t)$, etc.

Assuming that there are no correlations between different detector noises, we can readily write the probability of the multi-detector data set as the product of the probabilities for each detector's separate data set;

$$P(y_1, y_2, \cdots) = P(y_1) \cdot P(y_2) \cdots . \tag{26}$$

With this factorization, the generalized likelihood function of Eq. (25) takes on the form

$$\begin{aligned} \Lambda &= \frac{P(x^1|h) P(x^2|h) \cdots}{P(x^1|0) P(x^2|0) \cdots} \\ &= \frac{P(x^1|h)}{P(x^1|0)} \cdot \frac{P(x^2|h)}{P(x^2|0)} \cdots . \end{aligned} \tag{27}$$

But that means that the combined likelihood function for data from multiple detectors is simply the product of the likelihood functions for each detector's data set,

$$\Lambda = \Lambda^1 \times \Lambda^2 \times \cdots . \tag{28}$$

In consequence, the log of the general likelihood function (Eq. 25) is

$$\log \Lambda = \sum_{\rm X} \left[(x^{\rm X}|h^{\rm X}) - \frac{1}{2}(h^{\rm X}|h^{\rm X}) \right], \tag{29}$$

where the index 'x' runs over all the different detectors.

The important lesson is that absence of correlation implies linearity: As long as noises are uncorrelated, the logarithm of the likelihood ratio for the combined set can be computed by adding the logarithms of the likelihood ratio for each detector individually

Redefining the signal $h(t)$ (Eq. 6) for each individual detector 'x' as

$$h^{\rm X}(t) = \sum_{a=1}^{4} A^a h_a^{\rm X}, \tag{30}$$

and making this substitution in the likelihood function (Eq. 29), we have

$$\log \Lambda = \sum_{\rm X} (x^{\rm X}|A^a h_a^{\rm X}) - \frac{1}{2} \sum_{\rm X} (A^a h_a^{\rm X}|A^b h_b^{\rm X}). \tag{31}$$

Since the A^as are independent of the detectors, we can rewrite this as

$$\log \Lambda = A^a \sum_{\rm X} (x^{\rm X}|h_a^{\rm X}) - \frac{1}{2} A^a A^b \sum_{\rm X} (h_a^{\rm X}|h_b^{\rm X}), \tag{32}$$

with all the A^a in front.

It is helpful to generalize Eqs. (18) and (19) and introduce the quantities

$$\mathbf{H}_a = \sum_{\rm X} (x^{\rm X}|h_a^{\rm X}), \tag{33}$$

and

$$\mathbf{M}_{ab} = \sum_{\rm X} (h_a^{\rm X}|h_b^{\rm X}). \tag{34}$$

With their help, we can rewrite Eq. (32) as

$$\log \Lambda = A^a \mathbf{H}_a - \frac{1}{2} A^a A^b \mathbf{M}_{ab} \,. \tag{35}$$

By maximizing the log of likelihood function (Eq. 35), we obtain the general form of the $\mathcal{F}$-statistic, namely

$$\mathcal{F} = \frac{1}{2} \mathbf{H}_a \ \mathbf{M}^{-1^{ab}} \ \mathbf{H}_b \,. \tag{36}$$

Note that our vector $\mathbf{H}$ and matrix $\mathbf{M}$ in the above equation are the sum over the separate H and M, defined for each detector using Eqs. (18) and (19).

As Eq. (36) shows, the $\mathcal{F}$-statistic for multiple detectors has the same form as that for a single detector (cf. Eq. 24), except that each component represents a sum over all the detectors. In consequence, the recipe for calculating the combined $\mathcal{F}$-statistic is straightforward: compute the quantities H and M for each detector; add them to construct $\mathbf{H}$ and $\mathbf{M}$; combine these new quantities to obtain the $\mathcal{F}$-statistic.

Actually, following many other references, we will use $2\mathcal{F}$ instead of the $\mathcal{F}$ of Eq. (36); in other words, we work in terms of

$$2\mathcal{F} = \mathbf{H}_a \, \mathbf{M}^{-1^{ab}} \, \mathbf{H}_b \, . \tag{37}$$

In the situation under study here, both the observation time and 1 day [the time scale for variations of $a(t)$ and $b(t)$] are vastly larger than the period of the sought-for GWs (typically 10^{-2}–10^{-3}s). In consequence, we can replace $\cos^2\phi(t)$, $\sin^2\phi(t)$, $\cos\phi(t)\sin\phi(t)$ by their time average over one period, namely

$$\cos^2\phi(t) \rightarrow \frac{1}{2}\ , \ \ \sin^2\phi(t) \rightarrow \frac{1}{2}, \tag{38}$$

and

$$\cos\phi(t)\sin\phi(t) \rightarrow 0. \tag{39}$$

As a result,

$$(h_1|h_3) \approx 0, \quad (h_1|h_4) \approx 0, \quad (h_2|h_3) \approx 0, \quad (h_2|h_4) \approx 0. \tag{40}$$

We next define the amplitude coefficients $A,\ B$ and C as

$$\frac{1}{2}A^{\mathrm{X}} \equiv (h_1^{\mathrm{X}}|h_1^{\mathrm{X}}) = (h_3^{\mathrm{X}}|h_3^{\mathrm{X}}), \tag{41}$$

$$\frac{1}{2}B^{\mathrm{X}} \equiv (h_2^{\mathrm{X}}|h_2^{\mathrm{X}}) = (h_4^{\mathrm{X}}|h_4^{\mathrm{X}}), \tag{42}$$

$$\frac{1}{2}C^{\mathrm{X}} \equiv (h_1^{\mathrm{X}}|h_2^{\mathrm{X}}) = (h_3^{\mathrm{X}}|h_4^{\mathrm{X}}). \tag{43}$$

where the inner product is being defined as

$$(x|y) = \frac{2}{S_n(f_0)} \int_{t_0}^{t_p} x(t)y(t)\ dt, \tag{44}$$

with $S_n(f)$ to be the power spectral density of the signal. By using Eqs. (44) and (7-10), we can re-write the Eqs. (41-43) as

$$A^{\mathrm{X}} = \frac{2}{S_h^{\mathrm{X}}(f_0)} \int_{t_0}^{t_p} (a^{\mathrm{X}}(t))^2\ dt, \tag{45}$$

$$B^{\mathrm{X}} = \frac{2}{S_h^{\mathrm{X}}(f_0)} \int_{t_0}^{t_p} (b^{\mathrm{X}}(t))^2\ dt, \tag{46}$$

$$C^{\mathrm{X}} = \frac{2}{S_h^{\mathrm{X}}(f_0)} \int_{t_0}^{t_p} a^{\mathrm{X}}(t)b^{\mathrm{X}}(t)\ dt. \tag{47}$$

We use these quantities to define

$$A \equiv \sum_{\mathrm{X}} A^{\mathrm{X}}, \qquad B \equiv \sum_{\mathrm{X}} B^{\mathrm{X}}, \qquad C \equiv \sum_{\mathrm{X}} C^{\mathrm{X}}. \tag{48}$$

Note that in the definitions of $A^{\mathrm{X}}, B^{\mathrm{X}}$ and C^{X}, we have used the notation of CS[2], which differs from that of JKS[1] by a factor of $T/S_h(f)$, where T is the observation time.

Substituting the above quantities into the definition of Eq. (34), we obtain the generalized version of the matrix $\mathbf{M}$, namely

$$\mathbf{M} = \frac{1}{2}\begin{pmatrix} \mathcal{C} & \mathcal{O} \\ \mathcal{O} & \mathcal{C} \end{pmatrix}, \tag{49}$$

with

$$\mathcal{C} = \begin{pmatrix} A & C \\ C & B \end{pmatrix}, \quad \text{and} \quad \mathcal{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{50}$$

The inverse of matrix $\mathbf{M}$ is readily written down to be

$$\mathbf{M}^{-1} = \frac{2}{D}\begin{pmatrix} \mathcal{C}^{-1} & \mathcal{O} \\ \mathcal{O} & \mathcal{C}^{-1} \end{pmatrix}, \tag{51}$$

with

$$\mathcal{C}^{-1} = \begin{pmatrix} B & -C \\ -C & A \end{pmatrix}, \tag{52}$$

where $D \equiv AB - C^2$.

Returning to the general form of equation of $\mathcal{F}$-statistic (Eq. 37), we have

$$\begin{aligned} 2\mathcal{F} &= \mathbf{H}_a \, (\mathbf{M}^{-1})^{ab} \, \mathbf{H}_b \\ &= \frac{2}{D}\{B[(\mathbf{H}_1)^2 + (\mathbf{H}_3)^2] + A[(\mathbf{H}_2)^2 + (\mathbf{H}_4)^2] - 2C[\mathbf{H}_1\mathbf{H}_2 + \mathbf{H}_3\mathbf{H}_4]\}. \end{aligned} \tag{53}$$

If we define F_a and F_b as

$$2F_a \equiv \sqrt{(\mathbf{H}_1)^2 + (\mathbf{H}_3)^2} = \sum_{\mathrm{X}} (x^{\mathrm{X}} | h_1^{\mathrm{X}} - i h_3^{\mathrm{X}}) = 2\sum_{\mathrm{X}} F_a^{\mathrm{X}}, \tag{54}$$

$$2F_b \equiv \sqrt{(\mathbf{H}_2)^2 + (\mathbf{H}_4)^2} = \sum_{\mathrm{X}} (x^{\mathrm{X}} | h_2^{\mathrm{X}} - i h_4^{\mathrm{X}}) = 2\sum_{\mathrm{X}} F_b^{\mathrm{X}}, \tag{55}$$

where in addition

$$(h_1^{\mathrm{X}} - i h_3^{\mathrm{X}}) = a^{\mathrm{X}}(t) \mathrm{e}^{-i\phi^{\mathrm{X}}(t)}, \tag{56}$$

$$(h_2^{\mathrm{X}} - i h_4^{\mathrm{X}}) = b^{\mathrm{X}}(t) \mathrm{e}^{-i\phi^{\mathrm{X}}(t)}, \tag{57}$$

so that the result is

$$2F_a^{\mathrm{X}} = \frac{2}{S_h^{\mathrm{X}}(f_0)} \int_{t_0}^{t_p} x^{\mathrm{X}}(t) \; a^{\mathrm{X}}(t) \; \mathrm{e}^{-i\phi^{\mathrm{X}}(t)} \; dt, \tag{58}$$

$$2F_b^{\mathrm{X}} = \frac{2}{S_h^{\mathrm{X}}(f_0)} \int_{t_0}^{t_p} x^{\mathrm{X}}(t) \; b^{\mathrm{X}}(t) \; \mathrm{e}^{-i\phi^{\mathrm{X}}(t)} \; dt. \tag{59}$$

The $\mathcal{F}$-statistic can then be written as

$$2\mathcal{F} = \frac{8}{D}\{B|F_a^2| + A|F_b^2| - 2C\Re(F_a F_b^*)\}. \tag{60}$$

Note that, again, the multi-detector components of $\mathcal{F}$-statistic (Eq. 60), namely F_a, F_b and A, B, C are all simply *sums* over detector-specific quantities $F_a^{\mathrm{X}}, F_b^{\mathrm{X}}$ and $A^{\mathrm{X}}, B^{\mathrm{X}}, C^{\mathrm{X}}$, which can be calculated individually and then combined.

3 Discretization of the generalized $\mathcal{F}$-statistic's components (Generalized LALDemod)

In practice, data analysis is performed numerically with discrete data. This calls for a discretized expression for the $\mathcal{F}$-statistic derived in the previous section. The first technical note to derive such an expression for the case of a single detector, was done by Siemens [6]. We will briefly repeat his analysis here, generalizing the $\mathcal{F}$-statistic for multiple detectors; also, in [6], Siemens assumed that over the bandwidth of the signal $S_h(f)$ is constant and equal to $S_h(f_0)$ (just as JKS did), where f_0 is the frequency of the signal at $t = 0$. However while we will assume that $S_h(f)$ is constant in each SFT, it can differ from one SFT to the other.

We can write the discrete times of which the data is sampled as

$$t_{\alpha,j} = (\alpha - 1)T_{\text{sft}} + j\Delta t, \tag{61}$$

with $j = 1, ..., N$, and $\alpha = 1, .., M$ and $T_{\text{sft}} = N\Delta t$, where N is the number of time-steps per SFT, M is the number of SFTs and T_{sft} is the duration of each SFT.

As a first step towards discretization, let us calculate the Amplitude Modulation coefficients. Since the formalism is the same for all three coefficients A, B and C, we will give a detailed description for A, and only the results for B and for C. In Eq. (45), A was defined as

$$\begin{aligned} A &= 2\sum_{\mathrm{X}}(h_1^{\mathrm{X}}|h_1^{\mathrm{X}}) \\ &= 2\times\sum_{\mathrm{X}}\frac{2}{S_h^{\mathrm{X}}(f_0)}\int_{t_0}^{t_p} h_1^{\mathrm{X}}(t)h_1^{\mathrm{X}}(t)dt. \end{aligned} \tag{62}$$

Note that, by our assumption that the spectral density $S_h(f)$ of the noise is constant in each SFT, we can move it into the summation over different SFTs, and by using Eq. (7), we obtain

$$\begin{aligned} A &= 4\sum_{\mathrm{X}}\sum_{l=1}^{NM^{\mathrm{X}}}\frac{h_{1l}^{\mathrm{X}}\,h_{1l}^{\mathrm{X}}}{S_{h_l}^{\mathrm{X}}(f)}\,\Delta t \\ &= 4\sum_{\mathrm{X}}\sum_{\alpha=1}^{M^{\mathrm{X}}}\frac{a_\alpha^{\mathrm{X}}\,a_\alpha^{\mathrm{X}}}{S_{h_\alpha}^{\mathrm{X}}(f)}\sum_{j=1}^{N}\cos\phi_{\alpha j}^{\mathrm{X}}\cos\phi_{\alpha j}^{\mathrm{X}}\Delta t. \end{aligned} \tag{63}$$

Using the approximation given in Eq. (38), we have

$$\begin{aligned} \sum_{j=1}^{N}\cos\phi_j^{\mathrm{X}}\cos\phi_j^{\mathrm{X}}\Delta t &\approx \frac{N}{2}\Delta t \\ &= \frac{T_{\text{sft}}}{2}; \end{aligned} \tag{64}$$

inserting this expression into Eq. (63), the result is

$$
\begin{aligned}
A &\approx 2\,T_{\text{sft}} \sum_{\mathrm{X}} \sum_{\alpha=1}^{M^{\mathrm{X}}} \frac{(a^{\mathrm{X}}_{\alpha})^2}{S^{\mathrm{X}}_{h_\alpha}(f)} \\
&= 2\,T_{\text{sft}} \sum_{\mathrm{X}} \sum_{\alpha=1}^{M^{\mathrm{X}}} \left(\frac{a^{\mathrm{X}}_{\alpha}}{\sqrt{S^{\mathrm{X}}_{h_\alpha}(f)}} \right)^2 . \qquad (65)
\end{aligned}
$$

Note that T_{sft} is going to be fixed for data of different detectors. However, since we may have different amount of data for different detectors, the number of SFTs, M, can vary. Therefore, in the summation, we will give M an index x, indicating that it depends on the detector in question.

As was mentioned before, the B and C coefficients can be derived following the same procedure as for A. The results are

$$
B = 2\,T_{\text{sft}} \sum_{\mathrm{X}} \sum_{\alpha=1}^{M^{\mathrm{X}}} \left(\frac{b^{\mathrm{X}}_{\alpha}}{\sqrt{S^{\mathrm{X}}_{h_\alpha}(f)}} \right)^2 , \qquad (66)
$$

$$
C = 2\,T_{\text{sft}} \sum_{\mathrm{X}} \sum_{\alpha=1}^{M^{\mathrm{X}}} \left(\frac{a^{\mathrm{X}}_{\alpha}}{\sqrt{S^{\mathrm{X}}_{h_\alpha}(f)}} \frac{b^{\mathrm{X}}_{\alpha}}{\sqrt{S^{\mathrm{X}}_{h_\alpha}(f)}} \right) . \qquad (67)
$$

In order to simplify these expressions, we introduce the variables

$$
\hat{a}^{\mathrm{X}}_{\alpha} \equiv \left(\frac{a^{\mathrm{X}}_{\alpha}}{\sqrt{S^{\mathrm{X}}_{h_\alpha}(f)}} \right) , \qquad (68)
$$

$$
\hat{b}^{\mathrm{X}}_{\alpha} \equiv \left(\frac{b^{\mathrm{X}}_{\alpha}}{\sqrt{S^{\mathrm{X}}_{h_\alpha}(f)}} \right) , \qquad (69)
$$

as well as

$$
\hat{A}^{\mathrm{X}} \equiv \sum_{\alpha=1}^{M^{\mathrm{X}}} (\hat{a}^{\mathrm{X}}_{\alpha})^2 , \qquad (70)
$$

$$
\hat{B}^{\mathrm{X}} \equiv \sum_{\alpha=1}^{M^{\mathrm{X}}} (\hat{b}^{\mathrm{X}}_{\alpha})^2 , \qquad (71)
$$

$$
\hat{C}^{\mathrm{X}} \equiv \sum_{\alpha=1}^{M^{\mathrm{X}}} \hat{a}^{\mathrm{X}}_{\alpha}\, \hat{b}^{\mathrm{X}}_{\alpha} . \qquad (72)
$$

Using these variables, the Amplitude Modulation coefficients can be written as

$$
A = 2\,T_{\text{sft}} \sum_{\mathrm{X}} \hat{A}^{\mathrm{X}} , \qquad (73)
$$

$$
B = 2\,T_{\text{sft}} \sum_{\mathrm{X}} \hat{B}^{\mathrm{X}} , \qquad (74)
$$

$$
C = 2\,T_{\text{sft}} \sum_{\mathrm{X}} \hat{C}^{\mathrm{X}} . \qquad (75)
$$

The above expressions will give us the following form for the coefficient D:

$$\begin{aligned} D &= AB - C^2 \\ &= 4\ T_{SFT}^2 \sum_{\mathrm{X}} (\hat{A}^{\mathrm{X}} \hat{B}^{\mathrm{X}} - (\hat{C}^{\mathrm{X}})^2) \\ &= 4\ T_{SFT}^2 \sum_{\mathrm{X}} \hat{D}^{\mathrm{X}}. \end{aligned} \tag{76}$$

Next, we will calculate the quantities F_a and F_b, which depend on the detector orientations and positions. From Eq. (58), we derive the discretized version of F_a^{X},

$$F_a^{\mathrm{X}} = \sum_{\alpha=1}^{M^{\mathrm{X}}} \frac{a_\alpha^{\mathrm{X}}}{S_{h_\alpha}^{\mathrm{X}}(f)}\ \Delta t \left(\sum_{j=1}^{N} x_{\alpha,j}^{\mathrm{X}}(t) \mathrm{e}^{-i\phi_{\alpha,j}} \right), \tag{77}$$

where we have made the assumption that the duration of a SFT (in our case 30 minutes) is such that the amplitude modulation functions $a(t)$ and $b(t)$ (the quantities used in Eqs. (7) and (8) which have been defined in [1]) do not change significantly. We define the quantity inside parenthesis of Eq. (77) to be the SFT, which is the Fourier transform of our data for this short time-baseline ($N\Delta t$)

$$\tilde{x}_\alpha^{\mathrm{X}}(f) = \sum_{j=1}^{N} x_{\alpha j}^{\mathrm{X}}(t)\ \mathrm{e}^{-i2\pi f t_{\alpha,j}}. \tag{78}$$

Defining the general Discrete Fourier Transform (DFT) as

$$\tilde{x}_k = \sum_{j=0}^{N-1} x_j e^{-i2\pi jk/N}, \tag{79}$$

with its inverse

$$x_j = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{x}_k e^{i2\pi jk/N}, \tag{80}$$

we define the Discrete Fourier Transform (DFT) for this case as

$$\tilde{x}_{\alpha,k}^{\mathrm{X}}(f) = \sum_{j=1}^{N} x_{\alpha,j}^{\mathrm{X}}(t)\ \mathrm{e}^{-i2\pi kj/N}. \tag{81}$$

We translate between Eqs. (81) and (78) by equating the arguments of the exponential as

$$f_k t_{\alpha,j} = kj/N, \tag{82}$$

with $t_j = j\Delta t$ and $T_{\mathrm{sft}} = N\Delta t$, giving us that $f_k = k/N\Delta t$ and therefore $\Delta f = 1/T_{\mathrm{sft}}$, which is the frequency resolution of the data.

The inverse DFT of our data is then given by

$$x_{\alpha,j}^{\mathrm{X}}(t) = \frac{1}{N} \sum_{k=1}^{N} \tilde{x}_{\alpha,k}^{\mathrm{X}}(f)\ \mathrm{e}^{i2\pi kj/N}. \tag{83}$$

Substituting the $x^{\rm X}_{\alpha j}(t)$ into Eq. (77), we are led to

$$F_a^{\rm X} = \frac{\Delta t}{N}\sum_{\alpha=1}^{M^{\rm X}} \frac{a_\alpha^{\rm X}}{S_{h_\alpha}^{\rm X}(f)} \sum_{k=1}^{N} \tilde{x}^{\rm X}_{\alpha,k}(f) \sum_{j=1}^{N} \left({\rm e}^{(-i\phi_{\alpha,j} + i2\pi kj/N)} \right). \tag{84}$$

The arguments in the exponential relate to the evolution of the phase of our short segments of data (SFT). This segment is short enough that the phase evolution of the signal can be considered as linear within it. Therefore we can make a Taylor expansion of $\phi_{\alpha,j}$ around the middle of each segments (SFTs) as

$$\phi_{\alpha,j} = \phi_{\alpha,1/2} + \dot{\phi}_{\alpha,1/2}(t_{\alpha,j} - t_{\alpha,1/2}), \tag{85}$$

with $t_{\alpha,j}$ as defined in Eq. (61), so that

$$\phi_{\alpha,j} = \phi_{\alpha,1/2} + \dot{\phi}_{\alpha,1/2}(j - N/2)\Delta t = \phi_{\alpha,1/2} + \dot{\phi}_{\alpha,1/2}(j/N - 1/2)T_{\rm sft}. \tag{86}$$

Inserting this expression for the Taylor expansion into Eq. (84), we have

$$\begin{aligned} F_a^{\rm X} &= \frac{\Delta t}{N}\sum_{\alpha=1}^{M^{\rm X}} \frac{a_\alpha^{\rm X}}{S_{h_\alpha}^{\rm X}(f)} \sum_{k=1}^{N} \tilde{x}^{\rm X}_{\alpha,k}(f) \sum_{j=1}^{N} {\rm e}^{-i(\phi_{\alpha,1/2} + \dot{\phi}_{\alpha,1/2}(j/N-1/2)T_{\rm sft} - 2\pi kj/N)} \\ &= \frac{\Delta t}{N}\sum_{\alpha=1}^{M^{\rm X}} \frac{a_\alpha^{\rm X}}{S_{h_\alpha}^{\rm X}(f)} e^{-i(\phi_{\alpha,1/2} - \dot{\phi}_{\alpha,1/2}T_{\rm sft}/2)} \sum_{k=1}^{N} \tilde{x}^{\rm X}_{\alpha,k}(f) \sum_{j=1}^{N} {\rm e}^{-i(\dot{\phi}_{\alpha,1/2}T_{\rm sft} - 2\pi k)j/N}. \end{aligned} \tag{87}$$

We can simplify this further with the help of the following quantities:

$$\zeta \equiv \phi_{\alpha,1/2} - \dot{\phi}_{\alpha,1/2}T_{\rm sft}/2, \tag{88}$$

$$\varepsilon \equiv \dot{\phi}_{\alpha,1/2}T_{\rm sft} - 2\pi k. \tag{89}$$

Then Eq. (87) can be brought into the form

$$F_a^{\rm X} = \frac{\Delta t}{N}\sum_{\alpha=1}^{M^{\rm X}} \frac{a_\alpha^{\rm X}}{S_{h_\alpha}^{\rm X}(f)} e^{-i\zeta} \sum_{k=1}^{N} \tilde{x}^{\rm X}_{\alpha,k}(f) \sum_{j=1}^{N} {\rm e}^{-i\varepsilon j/N}. \tag{90}$$

Note that

$$\sum_{j=1}^{N} {\rm e}^{-i\varepsilon j/N} = \frac{1 - {\rm e}^{-i\varepsilon}}{1 - {\rm e}^{-i\varepsilon/N}}. \tag{91}$$

and that, furthermore, in the limit of very large N, the exponent in the denominator will be small, so that

$$\begin{aligned} \frac{1 - {\rm e}^{-i\varepsilon}}{1 - {\rm e}^{-i\varepsilon/N}} \approx \frac{1 - {\rm e}^{-i\varepsilon}}{1 - (1 - i\varepsilon/N)} &= \frac{iN}{\varepsilon}({\rm e}^{-i\varepsilon} - 1) \\ &= N(\frac{\sin\varepsilon}{\varepsilon} - i\frac{1-\cos\varepsilon}{\varepsilon}). \end{aligned} \tag{92}$$

Therefore,

$$F_a^{\rm X} \approx \Delta t \sum_{\alpha=1}^{M^{\rm X}} \frac{a_\alpha^{\rm X}}{S_{h_\alpha}^{\rm X}(f)} e^{-i\zeta} \sum_{k=1}^{N} \tilde{x}^{\rm X}_{\alpha,k}(f) P_{\alpha,k}, \tag{93}$$

where

$$P_{\alpha,k} = (\frac{\sin\varepsilon}{\varepsilon} - i\frac{1-\cos\varepsilon}{\varepsilon}) \tag{94}$$

is known as the Dirichlet kernel; it has a sharp peak around $\varepsilon = 0$ which, according to Eq. (89) would correspond to a sharp peak close to the value of the frequency index $k^* = \dot{\phi}_{\alpha,1/2}T_{\text{sft}}/2\pi$. This suggests a further expansion, this time around k^*; keeping only the lowest-order terms in Δk, the result for F_a as well as F_b is

$$F_a^{\mathrm{X}} \approx \Delta t \sum_{\alpha=1}^{M^{\mathrm{X}}} \frac{a_\alpha^{\mathrm{X}}}{S_{h_\alpha}^{\mathrm{X}}(f)} Q_\alpha \sum_{k=k^*-\Delta k}^{k^*+\Delta k} \tilde{x}_{\alpha,k}^{\mathrm{X}}(f) P_{\alpha,k}, \tag{95}$$

$$F_b^{\mathrm{X}} \approx \Delta t \sum_{\alpha=1}^{M^{\mathrm{X}}} \frac{b_\alpha^{\mathrm{X}}}{S_{h_\alpha}^{\mathrm{X}}(f)} Q_\alpha \sum_{k=k^*-\Delta k}^{k^*+\Delta k} \tilde{x}_{\alpha,k}^{\mathrm{X}}(f) P_{\alpha,k}, \tag{96}$$

where

$$Q_\alpha = \mathrm{e}^{-i\zeta}. \tag{97}$$

There is still one further step to take before we reach the final form of $F_{a,b}$. In the end, we are going to compute the $\mathcal{F}$-statistic numerically. As it stands, the quantities to be evaluated (and later, combined) in that calculation are of widely disparate orders of magnitude. The solution is to re-define the quantities needed for calculating the $\mathcal{F}$-statistic so that all of them are of order unity. The expectation value of input data, $x(t)$, has a value of order $\sim h_0 \sim 10^{-23}$; we will normalize it by the noise floor, resulting in a quantity of order unity. Using the Wiener-Khintchine theorem we can estimate power spectral density of the noise as

$$S_{h_\alpha}(f) \sim \frac{1}{T_{\text{sft}}} E[|\tilde{x}_\alpha|^2] \Longrightarrow \sqrt{S_{h_\alpha}(f)T_{\text{sft}}} \sim E[|\tilde{x}|], \tag{98}$$

where E denotes the expectation value. Therefore

$$E\left[\tilde{X}_\alpha(f)\right] = E\left[\frac{\tilde{x}_\alpha(f)}{\sqrt{S_{h_\alpha}(f)T_{\text{sft}}}}\right] \sim \mathcal{O}(1). \tag{99}$$

In consequence, we normalize the SFT as $\tilde{X}_{\alpha,k}^{\mathrm{X}}(f) = \tilde{x}_{\alpha,k}^{\mathrm{X}}(f)/\sqrt{S_{h_\alpha}^{\mathrm{X}}(f)T_{\text{sft}}}$. A correspondingly normalized version of F_a is

$$F_a^{\mathrm{X}} \approx \Delta t\sqrt{T_{\text{sft}}} \sum_{\alpha=1}^{M^{\mathrm{X}}} \frac{a_\alpha^{\mathrm{X}}}{\sqrt{S_{h_\alpha}^{\mathrm{X}}(f)}} Q_\alpha \sum_{k=k^*-\Delta k}^{k^*+\Delta k} \frac{\tilde{x}_{\alpha,k}^{\mathrm{X}}(f)}{\sqrt{S_{h_\alpha}^{\mathrm{X}}(f)T_{\text{sft}}}} P_{\alpha,k}. \tag{100}$$

Using Eq. (68) and taking $\tilde{X}'^{\mathrm{X}}_{\alpha,k}(f) = \Delta t \tilde{X}_{\alpha,k}^{\mathrm{X}}(f)$, we arrive at the final form of F_a,

$$F_a^{\mathrm{X}} \approx \sqrt{T_{\text{sft}}} \sum_{\alpha=1}^{M^{\mathrm{X}}} \hat{a}_\alpha^{\mathrm{X}}\, Q_\alpha \sum_{k=k^*-\Delta k}^{k^*+\Delta k} \tilde{X}'^{\mathrm{X}}_{\alpha,k}(f) P_{\alpha,k}. \tag{101}$$

Equivalently, F_b can be written as

$$F_b^{\mathrm{X}} \approx \sqrt{T_{\text{sft}}} \sum_{\alpha=1}^{M^{\mathrm{X}}} \hat{b}_\alpha^{\mathrm{X}}\, Q_\alpha \sum_{k=k^*-\Delta k}^{k^*+\Delta k} \tilde{X}'^{\mathrm{X}}_{\alpha,k}(f) P_{\alpha,k}. \tag{102}$$

Our data analysis software (which will be described in more detail in the following chapter) computes the components $P_{\alpha,k}$, $\tilde{X}'^{\mathrm{X}}_{\alpha,k}(f)$, Q_α, $\hat{a}_\alpha^{\mathrm{X}}$ and $\hat{b}_\alpha^{\mathrm{X}}$ and uses them to calculate $F_{a,b}$ and finally $\mathcal{F}$-statistic.

References

[1] P. Jaranowski, A. Królak, and B.F. Schutz (JKS), *Phys. Rev.* **D58**, 063001 (1998).

[2] Curt Cutler and Bernard F. Schutz, *Phys. Rev.* **D72**, 063006 (2005).

[3] P. Jaranowski, A. Królak, *Living Rev. Relativity* **8**,(2005), 3.

[4] M.H.A. Davis, "A review of Statistical Theory of Signal Detection", in B.F. Schutz, ed., *Gravitational Wave Data Analysis*, Proceeding of the NATO Advanced Research Woekshop, held at Dyffryn House, St. Nicholas, Cardiff, Wales, 6-9 July 1987, vol. **253** of NATO ASI Series C, 73-94.

[5] Bernard, F. Schutz, *Introduction to the Analysis of Low-Frequency Gravitational Wave Data*, gr-qc/9710080.

[6] Xavier Siemens, *Notes on LALDemod*, *LSC S2 Investigation page* (2004).