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Effects of using the wrong antenna pattern on sensitivity and parameter estimation

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Abstract

We show how the use of a wrong antenna pattern (detector response) affects detection sensitivity and parameter estimation for signal detection. We illustrate the general method on a simple example, and then apply it to the case of a cross-correlation search for an istropic stochastic gravitational-wave background.

1 Simple example

Let's take a simple example of a constant signal S modified by an antenna gain factor G_i and noise n_i (variance σ^2) to give an apparent strain h_i in an interferometer so that

$$h_i = G_i S + n_i \,, \tag{1}$$

where the subscript i represents the ith sample in time. Note the signal is constant but the gain varies with time due e.g., to Earth's rotation.

Conventionally we would determine the maximum likelihood estimator for S from

$$\chi^2 = \sum \frac{(h_i - G_i S)^2}{\sigma^2} \,. \tag{2}$$

Setting $d\chi^2/dS = 0$ we get the maximum likelihood estimator for the signal

$$\hat{S} = \frac{\sum h_i G_i}{\sum G_i^2} \,. \tag{3}$$

But what happens if we calculate this using the *wrong* antenna gain factor, W_i? We get an estimator that is

$$\hat{S}_W = \frac{\sum h_i W_i}{\sum W_i^2} \tag{4}$$

$$= \frac{\sum (G_i S + n_i) W_i}{\sum W_i^2} \tag{5}$$

$$= \frac{1}{\sum W_i^2} \left(S \sum G_i W_i + \sum W_i n_i \right) \,. \tag{6}$$

The expectation value for this estimator is

$$\langle \hat{S}_W \rangle = S \frac{\sum G_i W_i}{\sum W_i^2} \,. \tag{7}$$

Clearly, if $W_i = G_i$ we get the unbiased maximum-likelihood estimator of S. However, if $W_i \neq G_i$, there is a bias in the estimator given by

$$\text{bias} = \langle \hat{S}_W \rangle - S = S \left(\frac{\sum G_i W_i}{\sum W_i^2} - 1 \right) \,. \tag{8}$$

To see the effect of this on sensitivity, we need to calculate the variance of the estimator:

$$\text{var}(\hat{S}_W) = \left\langle \left(\hat{S}_W - \langle \hat{S}_W \rangle \right)^2 \right\rangle \tag{9}$$

$$= \left\langle \left(\frac{1}{\sum W_i^2} \left(S \sum G_i W_i + \sum W_i n_i \right) - S \frac{\sum G_i W_i}{\sum W_i^2} \right)^2 \right\rangle \tag{10}$$

$$= \left\langle \left(\frac{1}{\sum W_k^2} \right)^2 \sum W_i n_i \sum W_j n_j \right\rangle \tag{11}$$

$$= \left(\frac{1}{\sum W_k^2} \right)^2 \sum_{i,j} W_i W_j \langle n_i n_j \rangle \tag{12}$$

$$= \left(\frac{1}{\sum W_k^2} \right)^2 \sum_{i,j} W_i W_j \, \sigma^2 \, \delta_{ij} \tag{13}$$

$$= \frac{\sigma^2}{\sum W_i^2} \, . \tag{14}$$

The signal-to-noise ratio of the 'wrong' method is therefore

$$\mathrm{snr}_W = \frac{\langle \hat{S}_W \rangle}{\sqrt{\mathrm{var}(\hat{S}_W)}} \tag{15}$$

$$= \frac{S}{\sigma} \frac{\sum G_i W_i}{\sqrt{\sum W_i^2}} \, . \tag{16}$$

Again, if $W_i = G_i$ we get the 'best' signal-to-noise ratio

$$\mathrm{snr} = \frac{S}{\sigma} \sqrt{\sum G_i^2} \, . \tag{17}$$

The reduction in signal-to-noise ratio from using the wrong weights is therefore

$$r = 1 - \frac{\mathrm{snr}_W}{\mathrm{snr}} = 1 - \frac{\sum G_i W_i}{\sqrt{\sum G_i^2} \sqrt{\sum W_i^2}} \, , \tag{18}$$

where the last term is (sort of) the correlation coefficient between the two antenna gain factors.

2 Application to isotropic stochastic search

2.1 Cross-correlation statistic

Searches for an isotropic stochastic gravitational-wave background typically make use of the cross-correlation statistic, which can be written in the form:

$$Y = \frac{1}{T} \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df' \, \delta_T(f - f') K(f') h_1(f) h_2^*(f') \tag{19}$$

where

$$h_I(f) = s_I(f) + n_I(f) \tag{20}$$

are the Fourier transforms of the output of two detectors ($I = 1, 2$), and $K(f)$ is a filter function chosen to maximize the expected signal-to-noise ratio of Y. If one assumes (i) weak signals, (ii) uncorrelated detector noise $\langle n_1 n_2 \rangle = 0$, and (iii) an isotropic background with $\Omega_{\rm gw}(f) = \Omega_0 = {\rm const}$ (for which the associated gravitational-wave power is $H(f) = \Omega_0 f^{-3}$), then

$$K(f) = \mathcal{N} \frac{\gamma^*(f)|f|^{-3}}{P_1(|f|)P_2(|f|)}\,, \qquad \mathcal{N} = \left[\int_{-\infty}^{\infty} df\, \frac{|\gamma(f)|^2 f^{-6}}{P_1(|f|)P_2(|f|)} \right]^{-1} \tag{21}$$

where

$$\begin{aligned} \langle h_1(f) h_2^*(f') \rangle &= \langle s_1(f) s_2^*(f') \rangle && (22) \\ &= \delta(f - f')\gamma(f) H(f) && (23) \\ &= \delta(f - f')\gamma(f)\Omega_0 f^{-3}\,. && (24) \end{aligned}$$

Note that the numerators in $K(f)$ and the integrand of $\mathcal{N}$ are proportional to the complex conjugate of the (assumed) gravitational-wave cross-power and its absolute square, respectively.

Given the above definitions and assumptions on the stochastic signal model, one can show that

$$\langle Y \rangle = \Omega_0\,, \qquad {\rm var}(Y) = \frac{1}{4T} \left[\int_{-\infty}^{\infty} df\, \frac{|\gamma(f)|^2 f^{-6}}{P_1(|f|)P_2(|f|)} \right]^{-1}\,. \tag{25}$$

Thus, Y is an un-biased estimator of Ω_0 with optimal expected signal-to-noise ratio

$${\rm snr} = \Omega_0\, 2\sqrt{T} \left[\int_{-\infty}^{\infty} df\, \frac{|\gamma(f)|^2 f^{-6}}{P_1(|f|)P_2(|f|)} \right]^{1/2}\,. \tag{26}$$

In all of the above expressions, $h_I(f)$ are calibrated data (with units proportional to strain) and $P_I(f)$ are their corresponding power spectra (units proportional to strain^2).

2.2 Expressions in terms of uncalibrated data

To determine how the use of incorrect detector response functions change the above statistic, it is convenient to rewrite the relevant expressions above in terms of the *raw* (i.e., uncalibrated) detector output $r_I(f)$. Thus, let $R_I(f)$ denote the exact response functions that relate $r_I(f)$ and $h_I(f)$:

$$h_I(f) = R_I(f) r_I(f)\,. \tag{27}$$

Then one can rewrite Y, $K(f)$, and $\mathcal{N}$ as

$$Y = \frac{1}{T} \int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df'\, \delta_T(f - f') K(f') r_1(f) r_2^*(f') \tag{28}$$

where

$$K(f) = \mathcal{N} \frac{\Gamma^*(f)|f|^{-3}}{Q_1(|f|)Q_2(|f|)}\,, \qquad \mathcal{N} = \left[\int_{-\infty}^{\infty} df\, \frac{|\Gamma(f)|^2 f^{-6}}{Q_1(|f|)Q_2(|f|)}\right]^{-1} \tag{29}$$

where $Q_I(f)$ are the power spectra of $r_I(f)$,

$$P_I(f) = |R_I(f)|^2 Q_I(f) \tag{30}$$

and

$$\langle r_1(f) r_2^*(f')\rangle = \delta(f-f')\Gamma(f)\Omega_0 f^{-3} \tag{31}$$

is the associated cross-power. The overlap reduction functions $\gamma(f)$ and $\Gamma(f)$ are related by

$$\gamma(f) = R_1(f)R_2^*(f)\Gamma(f)\,. \tag{32}$$

2.3 Effect of using the wrong response function

Since the raw detector output and associated power spectra are prior to calibration, the effect of using a wrong detector response function enters only in the expression for $\Gamma(f)$. (There are also possible systematic errors that arise from the use of an incorrect signal model $H(f)$, but we will not consider that possibility here.) Thus, using an incorrect detector response in the expression for $\Gamma(f)$ leads to a modified cross-correlation statistic

$$Y_W = \frac{1}{T}\int_{-\infty}^{\infty} df \int_{-\infty}^{\infty} df'\, \delta_T(f-f') K_W(f') r_1(f) r_2^*(f') \tag{33}$$

where

$$K_W(f) = \mathcal{N}_W \frac{\Gamma_W^*(f)|f|^{-3}}{Q_1(|f|)Q_2(|f|)}\,, \qquad \mathcal{N}_W = \left[\int_{-\infty}^{\infty} df\, \frac{|\Gamma_W(f)|^2 f^{-6}}{Q_1(|f|)Q_2(|f|)}\right]^{-1}\,. \tag{34}$$

By repeating the same steps as above to calculate the expected value and variance of Y_W, one finds

$$\langle Y_W\rangle = \Omega_0 \frac{<\Gamma_W, \Gamma>}{<\Gamma_W,\Gamma_W>}\,, \qquad \mathrm{var}(Y_W) = \frac{1}{4T}\frac{1}{<\Gamma_W,\Gamma_W>} \tag{35}$$

where we've defined the inner product

$$<\Gamma_X,\Gamma_Y> = \int_{-\infty}^{\infty} df\, \frac{\Gamma_X^*(f)\Gamma_Y(f)}{f^6 Q_1(|f|)Q_2(|f|)} = 2\int_0^{\infty} df\, \frac{\mathrm{Re}\,[\Gamma_X^*(f)\Gamma_Y(f)]}{f^6 Q_1(f)Q_2(f)}\,. \tag{36}$$

The expected signal-to-noise ratio is

$$\mathrm{snr}_W = \Omega_0\, 2\sqrt{T}\frac{<\Gamma_W,\Gamma>}{\sqrt{<\Gamma_W,\Gamma_W>}}\,. \tag{37}$$

Thus, Y_W is a biased estimator of Ω_0, with

$$\text{bias} = \Omega_0 \left(\frac{< \Gamma_W, \Gamma >}{< \Gamma_W, \Gamma_W >} - 1 \right) \tag{38}$$

and reduced signal-to-noise ratio (relative to the optimal signal-to-noise ratio (26)):

$$r = 1 - \frac{\text{snr}_W}{\text{snr}} = 1 - \frac{< \Gamma_W, \Gamma >}{\sqrt{< \Gamma_W, \Gamma_W >}\sqrt{< \Gamma, \Gamma >}} \,. \tag{39}$$

Note that the above expressions for the bias and the reduction in signal-to-noise ratio for the cross-correlation statistic have the same form as those in Sec. 1 for the simple example, (8), (18) with the inner product $< \Gamma_W, \Gamma >$ playing the role of $\sum G_i W_i$.

2.4 Rewriting the inner product

To calculate the size of systematic errors given, for example, the nominal design sensitivity curves for *the* calibrated power spectra $P_I(f)$, it easiest to rewrite the inner product $< \Gamma_X, \Gamma_Y >$ in terms of the $P_I(f)$ and the associated overlap reduction functions $\gamma_X(f)$ and $\gamma_Y(f)$. Thus, assuming that the $P_I(f)$ are *properly* calibrated, we have

$$P_I(f) = |R_I(f)|^2 Q_I(f) \tag{40}$$

where $R_I(f)$ are the *exact* responses functions, as before. The overlap reduction functions, on the other hand, may be improperly calculated, e.g.,

$$\gamma_W(f) = R_{1W}(f) R^*_{2W}(f) \Gamma_W(f) \,. \tag{41}$$

where $R_I^W(f)$ are the response functions corresponding to whatever approximation is being made (e.g., a single cavity-pole approximation to the full Fabry-Perot response and the use of the long-wavelength antenna pattern functions). In terms of these response functions and the calibrated P's and γ's, one finds

$$< \Gamma_X, \Gamma_Y >= 2 \int_0^\infty df \, \frac{\text{Re}\left[\frac{|R_1|^2}{R^*_{1X} R_{1Y}} \frac{|R_2|^2}{R_{2X} R^*_{2Y}} \gamma^*_X(f) \gamma_Y(f) \right]}{f^6 P_1(f) P_2(f)} \,. \tag{42}$$

This can also be written in more compact form as

$$< \Gamma_X, \Gamma_Y >= 2 \int_0^\infty df \, \frac{\text{Re}\left[\bar{\gamma}^*_X(f) \bar{\gamma}_Y(f) \right]}{f^6 P_1(f) P_2(f)} \,, \tag{43}$$

where

$$\bar{\gamma}_X(f) = \frac{R_1(f)}{R_{1X}(f)} \frac{R^*_2(f)}{R^*_{2X}(f)} \gamma_X(f) \tag{44}$$

and similarly for $\gamma_Y(f)$. Note that systematic errors can enter the search through either an correct calibration $R_{IW}(f)$, or through an incorrect antenna pattern in $\gamma_W(f)$.