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Why elliptic orbits precess in the tabletop demonstration apparatus at the Hanford observatory

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1 Introduction

At LIGO's Hanford observatory, there is a device that is intended to demonstrate orbital motion in a gravitational potential. It is a large, black funnel fifty-five inches in diameter, and the idea is to roll ball bearings around in it and have them approximate planetary orbits. One thing you will notice, if you play around with it for a few minutes, is that highly elliptic orbits precess. There has been some casual speculation as to the source of this precession, with friction, imperfect leveling, and irregularities in the surface being proposed as possible causes [1]. In this technical note I will show that, even if the potential has a perfect hyperbolic ($1/r$) shape, the orbits will still precess due to the presence of a third (vertical) dimension, which is not present in ideal, two-body Kepler orbits.

2 Dynamics of a particle in a $1/r$-shaped funnel

The Lagrangian for a point particle is given by

$$\mathcal{L} = T - V$$

where V is the potential energy, and T is the kinetic energy. In cylindrical coordinates,

$$T = \frac{1}{2}m[\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2]$$

where the usual definitions apply, m is the mass, ρ is the radial coordinate, etc.

On earth, in a constant gravitational field, the potential is given by

$$V = mgz$$

For a particle confined to slide on the surface of a funnel, z will be constrained by the shape of the funnel's surface. Let's model the shape of the funnel by

$$z(\rho) = -\frac{\rho_0^2}{\rho}$$

so that the potential will look like that of a central gravitational force

$$V(\rho) = -\frac{mg\rho_0^2}{\rho}$$

Because z is constrained, we may also write $\dot{z}$ in terms of ρ.

$$\dot{z} = \left(\frac{\rho_0}{\rho}\right)^2 \dot{\rho}$$

The Lagrangian for the particle in the funnel is then

$$\mathcal{L} = \frac{1}{2}m\left\{\dot{\rho}^2\left[1+\left(\frac{\rho_0}{\rho}\right)^4\right]+\rho^2\dot{\phi}^2\right\}+\frac{mg\rho_0^2}{\rho} \tag{1}$$

Recall that the Lagrangian for a real gravitational potential of the same strength arising from a central force would be

$$\mathcal{L} = \frac{1}{2}m\left\{\dot{\rho}^2+\rho^2\dot{\phi}^2\right\}+\frac{mg\rho_0^2}{\rho} \tag{2}$$

The difference is that, with the central force, the orbits are confined to a plane, whereas with the funnel, the particle has motion in the z-direction. This added vertical motion, even though it is constrained to the surface of the funnel, gives rise to the term $(1+(\rho_0/\rho)^4)$ that multiplies the $\dot{\rho}^2$ term in Equation 1. This term is nearly 1 when $\rho \gg \rho_0$ (*i.e.* when the particle is far from the central hole, and the funnel is shallow), and in this case the dynamics closely approximate those of the central force law.

The orbital equations for the particle in a funnel are

$$\ddot{\phi} = -2\frac{\dot{\rho}\dot{\phi}}{\rho} \tag{3}$$

and

$$\ddot{\rho} = \frac{\rho\dot{\phi}^2 - g\left(\frac{\rho_0}{\rho}\right)^2 + 2\left(\frac{\rho_0}{\rho}\right)^4\frac{\dot{\rho}^2}{\rho}}{1+\left(\frac{\rho_0}{\rho}\right)^4} \tag{4}$$

These are obtained from the usual procedure for a generalized coordinate q,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \frac{\partial \mathcal{L}}{\partial q} = 0$$

We can use the same procedure for finding the equations of motion for the central force law, where the orbits are confined to the plane of the ecliptic.

$$\ddot{\phi} = -2\frac{\dot{\rho}\dot{\phi}}{\rho} \tag{5}$$

and

$$\ddot{\rho} = \rho\dot{\phi}^2 - g\left(\frac{\rho_0}{\rho}\right)^2 \tag{6}$$

The dynamics of the particle in the funnel, Equations 3 and 4, reduce to those of the central force law, Equations 5 and 6, for very large orbits ($\rho \gg \rho_0$), as we expect. Somewhat surprisingly, though, the funnel dynamics become identical to the central-force-law dynamics for circular orbits, where $\dot{\rho} = \ddot{\rho} = 0$.

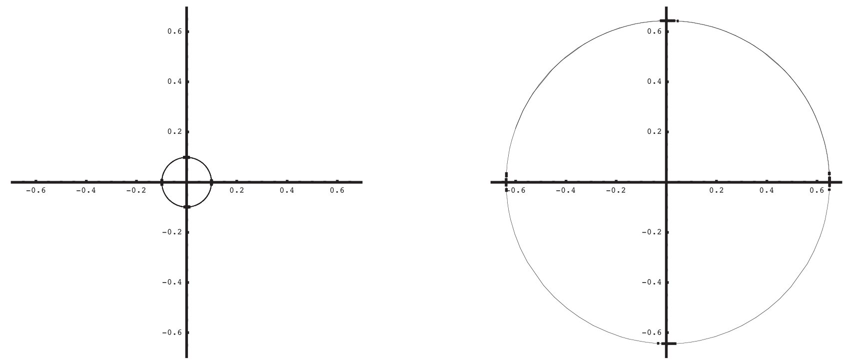

Figure 1: Parametric plot of the numerical solutions to the orbital equations of motion for small, circular orbits (left) and for large, nearly circular orbits (right). The equations of motion for both the central force law and the particle in a funnel produced orbits that were visually indistinguishable and closed.

3 Orbits

I used *Mathematica* to numerically solve Equations 3 and 4, and the central-force-law Equations 5 and 6. I looked at three different cases:

1. Large ($\rho \gg \rho_0$) orbits, nearly but not quite circular.
2. Small ($\rho \leq \rho_0$), circular orbits.
3. Highly elliptic orbits where the perihelion (distance of closest approach) is comparable to ρ_0.

In each case I used

$$\rho_0 = 0.15\text{m}$$

and

$$g = 9.81\frac{\text{m}}{\text{s}^2}$$

Results are shown in the figures. Sure enough, the highly elliptic orbits with tight perihelions precessed for the funnel dynamics, but not for the central force law. Also as expected, neither the large orbits nor the small, circular orbits exhibited any precession.

4 Conclusions

It is a well-known result from classical dynamics that the only central force laws that can produce closed orbits for all bound particles are the inverse square law ($1/r$ potential) and Hooke's law (r potential) [2], and this result is known as Bertrand's theorem [3]. Orbital precession is not unique to general relativity. Any deviation from an inverse-r potential can cause it. In fact, less than ten percent of Mercury's vaunted perihelion precession is due to general relativistic effects, the rest being caused

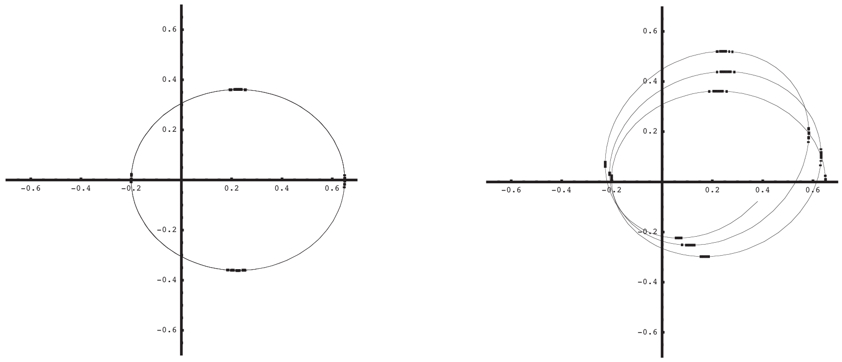

Figure 2: Highly elliptic orbits for the central force law (left) and the funnel (right). The extra term in the kinetic energy for the particle in a funnel, due to the vertical degree of freedom, causes the orbit to precess.

by the gravitational pull of the other planets and the fact that the sun is not perfectly spherical [4]. With this in mind, it should not be surprising that you can produce perihelion precession in a simple demonstration apparatus. What is surprising, to me at least, is that you can do so *even if the potential has the correct* $1/r$ *form*. The precession in this case comes from the vertical term in the kinetic energy, which is the *only* thing that is different between the funnel equations of motion and those of a central gravitational force. No deviation from a $1/r$ potential is necessary, nor is there any need to take into account friction, table tilt, bumps, or the rolling of the ball.

References

[1] Fred Raab, *Private communication*

[2] J. Bertrand, *Comptes Rendus* 77, 849-853 (1873).

[3] Herbert Goldstein, *Classical Mechanics*, Second Edition, Addison-Wesley (1980).

[4] Hans Ohanian and Remo Ruffini, *Gravitation and Spacetime*, Second Edition, W.W. Norton and Company, New York (1994).