PowerFlux polarization analysis
(draft, pre-alpha)

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Contents

1 Introduction

During the course of analysis PowerFlux generates estimates of excess power for several (usually 4) fixed linear polarizations. The following text is a presentation of method to translate these limits into limits on arbitrary polarization.

2 Single SFT

First, we consider a single SFT.

Suppose we have an elliptically polarized source with major axis tilted at an angle ψ w.r.t. vertical axis.

We assume that during SFT period (normally 30 minutes) the frequency of source can be assumed constant.

$$\begin{aligned} h'_+ &= A_+ \cos(\omega t) \\ h'_\times &= A_\times \sin(\omega t) \end{aligned}$$

- For linear polarization we set $A_+ = h_0^{LIN}$, $A_\times = 0$.
- For circular polarization we set $A_+ = A_\times = h_0^{CIR}$.
- A pulsar signal would be described as $A_+ = h_0 \left(1 + \cos^2(\iota)\right)/2$, $A_\times = h_0 \cos(\iota)$, with $h_0 = A_+ + \sqrt{A_+^2 - A_\times^2}$ and $\cos(\iota) = A_\times / \left(A_+ + \sqrt{A_\times^2 - A_+^2}\right)$

In order to describe the influence of the incoming signal on the values computed for a fixed linear polarization we introduce a coordinate system rotated at an angle α. In it we have:

$$\begin{aligned} h_+ &= A_+ \cos(\omega t)\cos(\epsilon) - A_\times \sin(\omega t)\sin(\epsilon) \\ h_\times &= A_+ \cos(\omega t)\sin(\epsilon) + A_\times \sin(\omega t)\cos(\epsilon) \end{aligned}$$

where we use $\epsilon = 2(\psi - \alpha)$.

The average detected signal power during SFT period is:

$$\langle P_{sig} \rangle = \langle (F_+ h_+ + F_\times h_\times)^2 \rangle$$

where $F_\times$ and F_+ is the amplitude response.

After substitution[1] we arrive at

$$\langle P_{sig} \rangle = \frac{1}{4}\left[(F_+^2+F_\times^2)(A_+^2+A_\times^2) + (F_+^2-F_\times^2)(A_+^2-A_\times^2)\cos(2\epsilon) + 2F_+F_\times(A_+^2-A_\times^2)\sin(2\epsilon) \right]$$

- For circular polarization $\langle P_{sig} \rangle = \frac{1}{2}\left(F_+^2+F_\times^2\right)\left(h_0^{CIR}\right)^2$

3 Weighted power average

During signal analysis PowerFlux constructs a weighted power average:

$$\bar{x}(\alpha) = \frac{\sum w_i \frac{P^{DET}(i)}{F_+^2(\alpha,i)}}{\sum w_i}$$

where $P^{DET}(i)$ is the power measured by the detector, $F_+^2(\alpha, i)$ is the amplitude response to linearly polarized signal with parameter α.

The weights w_i are usually chosen as

$$w_i = \frac{F_+^4(\alpha,i)}{\mathrm{TMedian}(i)^2}$$

where TMedian is an estimate of average power in an SFT (for more detail see PowerFlux algorithms and implementation document).

We can expand $P^{DET}(i)$ as

$$P^{DET}(i) = \langle P_{SIG} \rangle_i + \langle n(t)^2 \rangle + 2\langle P_{SIG} n(t) \rangle$$

where $n(t)$ is the noise present in the detector.

We assume that the last term vanishes with the averaging.

Thus

$$\bar{x}(\alpha) = \bar{n}(\alpha) + \frac{1}{4}\left[(1+\beta_2)(A_+^2+A_\times^2) + (1-\beta_2)(A_+^2-A_\times^2)\cos(2\epsilon) + 2\beta_1(A_+^2-A_\times^2)\sin(2\epsilon) \right] \tag{1}$$

where we use

$$\begin{aligned} \beta_1 &:= \frac{\sum_i w_i F_\times/F_+}{\sum_i w_i} \\ \beta_2 &:= \frac{\sum_i w_i F_\times^2/F_+^2}{\sum_i w_i} \end{aligned}$$

Note that the constants β_1 and β_2 depend implicitly on α through F_+ and $F_\times$.

[1] TODO: add the algebra for verification

- Linear polarization:

$$\bar{x}(\alpha) = \bar{n}(\alpha) + \frac{1}{4}(h_0^{LIN})^2 \left[1 + \beta_2 + (1-\beta_2)\cos(2\epsilon)\right]$$

- Circular polarization:

$$\bar{x}(\alpha) = \bar{n}(\alpha) + \frac{1}{2}(h_0^{CIR})^2(1 + \beta_2)$$

- Arbitrary pulsar: TODO

4 Betas for arbitrary sampled polarization

A new additon to PowerFlux is the ability to sample arbitrary elliptic polarization, in particular, PowerFlux now samples circular polarization by default. Thus we need to work out an analog of formula 1 and extend β constants to this case.

Suppose we have an elliptic polarization with amplitude response $\epsilon_+ F_+^2 + \epsilon_\times F_\times^2$ assuming appropriate orientation angle ψ for F_+ and $F_\times$.

We derive

$$\begin{aligned} \bar{x}(\alpha) \quad = \quad & \bar{n}(\alpha) + \tfrac{1}{4}\big[(a_1 + a_2\beta_2)(A_+^2 + A_\times^2) + \\ & (b_1 + b_2\beta_2)(A_+^2 - A_\times^2)\cos(2\epsilon) + \\ & + 2\beta_1(A_+^2 - A_\times^2)\sin(2\epsilon)\big] \end{aligned} \tag{2}$$

where we use

$$\begin{aligned} \beta_1 \quad &:= \quad \tfrac{1}{\sum_i w_i}\sum_i w_i \tfrac{F_\times F_+}{\epsilon_+ F_+^2 + \epsilon_\times F_\times^2} \\ \beta_2 \quad &:= \quad \tfrac{1}{\sum_i w_i}\sum_i w_i \tfrac{-\epsilon_\times F_+^2 + \epsilon_+ F_\times^2}{\epsilon_+ F_+^2 + \epsilon_\times F_\times^2} \end{aligned}$$

Note that for the case $\epsilon_+ = 1$ and $\epsilon_\times = 0$ we have our original definition.

The constants a_1, a_2, b_1 and b_2 must satisfy:

$$\begin{cases} a_1\epsilon_+ - a_2\epsilon_\times = 1 \\ a_1\epsilon_\times + a_2\epsilon_+ = 1 \end{cases}$$

and

$$\begin{cases} b_1\epsilon_+ - b_2\epsilon_\times = 1 \\ b_1\epsilon_\times + b_2\epsilon_+ = -1 \end{cases}$$

The solution to these equations always exists as long as one of ϵ_+ or $\epsilon_\times$ is non-zero, which always holds.

$$\begin{aligned} a_1 &= \tfrac{\epsilon_+ + \epsilon_\times}{\epsilon_+^2 + \epsilon_\times^2} \\ a_2 &= \tfrac{\epsilon_+ - \epsilon_\times}{\epsilon_+^2 + \epsilon_\times^2} \\ b_1 &= \tfrac{\epsilon_+ - \epsilon_\times}{\epsilon_+^2 + \epsilon_\times^2} \\ b_2 &= \tfrac{-\epsilon_+ - \epsilon_\times}{\epsilon_+^2 + \epsilon_\times^2} \end{aligned}$$

Substituting into 2 we obtain the analog of formula 1:

$$\begin{aligned} \bar{x}(\alpha) \;=\; & \bar{n}(\alpha) + \tfrac{1}{4}\Big[\tfrac{\epsilon_+ + \epsilon_\times + (\epsilon_+ - \epsilon_\times)\beta_2}{\epsilon_+^2 + \epsilon_\times^2}(A_+^2 + A_\times^2) + \\ & \tfrac{\epsilon_+ - \epsilon_\times - (\epsilon_+ + \epsilon_\times)\beta_2}{\epsilon_+^2 + \epsilon_\times^2}(A_+^2 - A_\times^2)\cos(2\epsilon) + \\ & +2\beta_1(A_+^2 - A_\times^2)\sin(2\epsilon)\Big] \end{aligned} \tag{3}$$

- Linear polarization:

$$\bar{x}(\alpha) = \bar{n}(\alpha) + \frac{1}{4}(h_0^{LIN})^2\left[\frac{\epsilon_+ + \epsilon_\times + (\epsilon_+ - \epsilon_\times)\beta_2}{\epsilon_+^2 + \epsilon_\times^2} + \frac{\epsilon_+ - \epsilon_\times - (\epsilon_+ + \epsilon_\times)\beta_2}{\epsilon_+^2 + \epsilon_\times^2}\cos(2\epsilon)\right]$$

- Circular polarization:

$$\bar{x}(\alpha) = \bar{n}(\alpha) + \frac{1}{2}(h_0^{CIR})^2\frac{\epsilon_+ + \epsilon_\times + (\epsilon_+ - \epsilon_\times)\beta_2}{\epsilon_+^2 + \epsilon_\times^2}$$

- Arbitrary pulsar: TODO