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FDT approach calculations of brownian noise in thin layer

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Here we present the detail calculations the brownian noise in coating for gaussian spot (see also [1]) using calculations[2] of thermoelastic noise.

[Note added by Kip S. Thorne: This is an unpublished manuscript written by Vyatchanin in 2004, as part of a dialogue between him, me, and Richard O'Shaughnessy, over how thermal noise scales with the size and shape of the laser beam. The key result in this manuscript is the equation in Section V.]

I. INTRODUCTION

We consider the infinite half-space covered by thin layer with thickness d. The parameters of film we denote by subscript $_f$ or superscript $^{(f)}$ and parameters of substrate — by subscript $_s$ or superscript $^{(s)}$. ν_f and ν_s are Poison ratios, E_f, E_s are Young modula.

We are interesting in fluctuations of generalized positions $\bar X(t)$ of surface: it is averaged over the beam spot's Gaussian power profile normal displacement u_z of the surface:

$$\bar X \; = \int \frac{e^{-r^2/r_0^2}}{\pi r_0^2}\, u_z(\vec r)\, d\vec r\, . \tag{1}$$

Here integral is over the surface, r_0 is the radius at which the spot's light power flux has dropped to $1/e$ of its central value.

Scheme of calculations. Brownian (structural) noise can be computed using fluctuation-dissipation theorem [3, 4] by the following thought experiment. We imagine applying a sinusoidally oscillating pressure,

$$P \equiv P(r) = F_0 \frac{e^{-r^2/r_0^2}}{\pi r_0^2}\, e^{i\omega t} \tag{2}$$

to face of half infinite space (covered by layer). Here F_0 is a constant force amplitude, ω is the angular frequency at which one wants to know the spectral density of thermal noise, and the pressure distribution (2) has precisely the same spatial profile as that of the generalized coordinate

$\bar X$, whose thermal noise $S_{\bar X}(f)$ one wishes to compute.

The oscillating pressure P produces elastic energy in half space, where it gets dissipated. Computing the rate of this energy dissipation, $W_{\rm diss}$, averaged over the period $2\pi/\omega$ of the pressure oscillations we can just write down (in according with fluctuation-dissipation theorem) the spectral density of the noise $S_{\bar X}(\omega)$:

$$S_{\bar X}(\omega) = \frac{8 k_B T W_{\rm diss}}{F_0^2 \omega^2} \tag{3}$$

here k_B is Boltzman's constant.

The rate $W_{\rm diss}$ of dissipation via structural losses is given by the following standard expression:

$$W_{\rm diss} = \omega\phi\mathcal{E}, \quad \mathcal{E} = \left\langle \int P(r)\, u_z\, d\vec r \right\rangle = \frac{1}{2}\int P(r)\, u_z\, d\vec r \tag{4}$$

Here $\mathcal{E}$ is averaged elastic energy, which is equal to work of external pressure force. The integral is taken over surface of half space; ϕ is loss angle, $\langle ...\rangle$ denotes an average over the pressure's oscillation period $1/f = 2\pi/\omega$ (in practice it gives just a simple factor $\langle(\Re\, e^{i\omega t})^2\rangle = 1/2$).

The computation below is made fairly simple by quasistatic approximations [7]: we can approximate the oscillations of stress and strain in the test mass, induced by the oscillating pressure P, as *quasistatic.* This approximation permits us, at any moment of time t, to compute the displacement field $\vec u$ from the equations of static stress balance (equation (7.4) in [5])

$$(1-2\nu)\nabla^2 \vec u + \vec\nabla(\vec\nabla\cdot\vec u) = 0 \tag{5}$$

with the boundary condition that the normal pressure on face be $P(r,t)$ (2) and that all other non-tangential stresses vanish at the surface.

We can devide value of $W_{\rm diss}$ into substrate and coating (film) parts:

$$(6)$$

$$W^f_{\rm diss} = W^f_\perp + W^f_\|, \quad W^f_\perp = \frac{\omega\phi\, d}{2}\int P(r)\, u^f_{zz}\, d\vec r, \quad W^f_\| = \frac{\omega\phi}{2}\int \big(\sigma_{xz}u_x + \sigma_{yz}u_y\big)\, d\vec r, The original of this message bounced back, d$$

$$(7)$$

Below we calculate the elastic problem in substrate and film separately.

II. SUBSTRATE (ELASTIC INFINITE HALF SPACE)

We can assume that layer does not influence on deformations in substrate due to its small thickness. Then we can use the solution to the quasistatic stress-balance equation (5) given by a Green's-function expression (see (8.18) in [5]) with $F_x = F_y = 0$, $F_z = P(r)$, integrated over the surface of the test mass:

$$u_x(x,y,z) = \frac{(1+\nu_s)}{2\pi E_s}\frac{F_0}{\pi r_0^2}\int_{-\infty}^{\infty} dx'\, dy'\, e^{-(x'^2+y'^2)/r_0^2}(x-x')\left\{\frac{z}{r^3} - \frac{(1-2\nu_s)}{r(r+z)}\right\}, \tag{8}$$

$$u_y(x,y,z) = \frac{(1+\nu_s)}{2\pi E_s}\frac{F_0}{\pi r_0^2}\int_{-\infty}^{\infty} dx'\, dy'\, e^{-(x'^2+y'^2)/r_0^2}(y-y')\left\{\frac{z}{r^3} - \frac{(1-2\nu_s)}{r(r+z)}\right\}, \tag{9}$$

$$u_z(x,y,z) = \frac{(1+\nu_s)}{2\pi E_s}\frac{F_0}{\pi r_0^2}\int_{-\infty}^{\infty} dx'\, dy'\, e^{-(x'^2+y'^2)/r_0^2}\left\{\frac{2(1-\nu_s)}{r} + \frac{z^2}{r^3}\right\}, \tag{10}$$

$$r = \sqrt{(x-x')^2 + (y-y')^2 + z^2}.$$

Note that we can not put $z = 0$ here due to formal divergence of surface integral at $z = 0$.

Below we will need the expression for $\Theta^{(s)} = \textbf{div}\,\vec u$. Using (8-10) one can calculate longitudial and transversal parts of $\Theta^{(s)}$ separately:

$$\Theta^{(s)}_\| = \partial_x u_x + \partial_y u_y = \frac{(1+\nu_s)}{2\pi E_s}\frac{F_0}{\pi r_0^2}\int_{-\infty}^{\infty} dx'\, dy'\, e^{-(x'^2+y'^2)/r_0^2}\left(\frac{3z^3}{r^5} - \frac{2(1-\nu_s)z}{r^3}\right)$$

$$\Theta^{(s)}_\perp = \partial_z u_z = \frac{(1+\nu_s)}{2\pi E_s}\frac{F_0}{\pi r_0^2}\int_{-\infty}^{\infty} dx'\, dy'\, e^{-(x'^2+y'^2)/r_0^2}\left(\frac{2\nu_s z}{r^3} - \frac{3z^3}{r^5}\right)$$

$$\Theta_s = \Theta^{(s)}_\| + \Theta^{(s)}_\perp = -\frac{(1+\nu_s)(1-2\nu_s)F_0}{\pi^2 r_0^2 E_s}\int_{-\infty}^{\infty} dx'\, dy'\, e^{-(x'^2+y'^2)/r_0^2}\left(\frac{z}{r^3}\right)$$

Using result of calculations presented in Appendix A we obtain

$$\Theta_s = -\frac{(1+\nu_s)(1-2\nu_s)F_0}{2\pi^2 E_s}\, e^{i\omega t}\iint_{-\infty}^{+\infty} e^{-k_\perp^2 r_0^2/4} e^{-k_\perp z} e^{i(k_x x + k_y y)}\, dk_x\, dk_y, \tag{11}$$

$$k_\perp \equiv \sqrt{k_x^2 + k_y^2},$$

$$\Theta^{(s)}_\| = \frac{(1+\nu_s)F_0}{4\pi^2 E_s}\, e^{i\omega t}\iint_{-\infty}^{+\infty} e^{-k_\perp^2 r_0^2/4} e^{-k_\perp z} e^{i(k_x x + k_y y)}\Big(k_\perp z - 1 + 2\nu_s\Big)\, dk_x\, dk_y, \tag{12}$$

$$\Theta^{(s)}_\perp = \frac{(1+\nu_s)F_0}{4\pi^2 E_s}\, e^{i\omega t}\iint_{-\infty}^{+\infty} e^{-k_\perp^2 r_0^2/4} e^{-k_\perp z} e^{i(k_x x + k_y y)}\Big(2\nu_s - k_\perp z - 1\Big)\, dk_x\, dk_y, \tag{13}$$

$$\Theta_s|_{z=0} = -\frac{2(1+\nu_s)(1-2\nu_s)P}{E_s}, \tag{14}$$

$$\Theta^{(s)}_\||_{z=0} = -\frac{(1+\nu_s)(1-2\nu_s)\,P}{E_s}, \tag{15}$$

$$\Theta_\perp^{(s)}|_{z=0} \;=\; -\frac{(1+\nu_s)(1-2\nu_s)\,P}{E_s} \tag{16}$$

The formulas above for the particular case $z=0$ may be obtained easier using formulas pointed by Kip Thorne:

$$\lim_{z\to 0}\frac{z}{\left(\sqrt{x^2+y^2+z^2}\right)^3}=\frac{\delta(x)\,\delta(y)}{2\pi},\qquad \lim_{z\to 0}\frac{z^3}{\left(\sqrt{x^2+y^2+z^2}\right)^5}=\frac{\delta(x)\,\delta(y)}{6\pi}. \tag{17}$$

A. Cross Term

Here we calculate u_{xy} using (8 and 17):

$$u_{xy}(x,y,z)|_{z\to 0} \;=\; \frac{(1+\nu_s)}{2\pi E_s}\,\frac{F_0}{\pi r_0^2}\int_{-\infty}^{\infty} dx'\,dy'\,e^{-(x'^2+y'^2)/r_0^2}\,\frac{(x-x')(y-y')}{r^2}\left\{\frac{-3z}{r^3}+\frac{1-2\nu_s}{(r+z)}\left[\frac{1}{r}+\frac{1}{r+z}\right]\right\}= \tag{18}$$

$$=\; \frac{(1+\nu_s)}{2\pi E_s}\,\frac{F_0}{\pi r_0^2}\int_{-\infty}^{\infty} dx'\,dy'\,e^{-(x'^2+y'^2)/r_0^2}\,\frac{(x-x')(y-y')}{r^2}\,\frac{1-2\nu_s}{(r+z)}\left[\frac{1}{r}+\frac{1}{r+z}\right]= \tag{19}$$

$$=\; \frac{(1+\nu_s)}{2\pi E_s}\,\frac{F_0}{\pi r_0^2}\int_{-\infty}^{\infty} \frac{dk_x\,dk_y}{(2\pi)^2}\,e^{-(k_x^2+k_y^2)r_0^2/4}\,e^{-ik_x x-ik_y y}\times I_{xy}, \tag{20}$$

$$I_{xy} \;=\; \int_{-\infty}^{\infty} dx'\,dy'\,e^{ik_x(x-x')+ik_y(y-y')}\,\frac{(x-x')(y-y')}{r^2}\,\frac{1-2\nu_s}{(r+z)}\left[\frac{1}{r}+\frac{1}{r+z}\right]= \tag{21}$$

$$=\; \int r\,dr\,d\phi\,e^{ik_\perp r\cos(\phi-\phi_0)}\,\sin\phi\cos\phi\,\frac{1-2\nu_s}{(r+z)}\left[\frac{1}{r}+\frac{1}{r+z}\right],\quad k_x=k_\perp\cos\phi_0,\quad k_y=k_\perp\sin\phi_0,$$

It seems that the result of calculation does not have to depend on ϕ_0 due to axial simmetry of pressure profile. Hence one can assume $\phi_0=0$. Then integrating over ϕ we obtain that $I_{xy}=0$.

III. LAYER (FILM)

We assume that deformations of layer in transversal plane are the same as in substrate, i.e. $\Theta_\parallel^{(f)}=\Theta_\parallel^{(s)}|_{z=0}$. One can use equation (5.13) for stress in [5] for calculation $\Theta_\perp^{(f)}\equiv u_{zz}^{(f)}$:

$$\sigma_{zz}\;\equiv\;-P=\frac{E_f}{(1+\nu_f)(1-2\nu_f)}\Big((1-\nu_f)u_{zz}^{(f)}+\nu_f\,\underbrace{(u_{xx}^{(f)}+u_{yy}^{(f)})}_{\Theta_\parallel^{(s)}|_{z=0}}\Big). \tag{22}$$

Using this equation one can find $u_{zz}^{(f)}$ and full expansion Θ_f in layer introducing "effective" modula Y_f and Y_s:

$$u_{zz}^{(f)} \;=\; -\frac{P}{Y_f(1-\nu_f)}\left(1-\frac{\nu_f Y_f}{Y_s}\right), \tag{23}$$

$$\Theta_f \;=\; -\frac{P}{Y_s}\left(1+\frac{Y_s}{(1-\nu_f)Y_f}-\frac{\nu_f}{1-\nu_f}\right)=-\frac{P}{Y_f(1-\nu_f)}\left(1+\frac{Y_f(1-2\nu_f)}{Y_s}\right), \tag{24}$$

$$Y_s \;=\; \frac{E_s}{(1+\nu_s)(1-2\nu_s)},\quad Y_f=\frac{E_f}{(1+\nu_f)(1-2\nu_f)}$$

IV. SPECTRAL DENSITY

Now we can calculate the spectral density of brownian noise in coating using (23):

$$W_{\rm diss}^f=\frac{\omega\phi\,d}{2}\times\frac{1}{Y_f(1-\nu_f)}\left(1-\frac{\nu_f Y_f}{Y_s}\right)\times\int P^2 d\vec r=\frac{\omega\phi\,d}{2}\times\frac{F_0^2}{2\pi r_0^2}\times\frac{1}{Y_f(1-\nu_f)}\left(1-\frac{\nu_f Y_f}{Y_s}\right), \tag{25}$$

$$S_X^f(\omega)=\frac{2k_B T\,\phi\,d}{\pi r_0^2\,\omega}\times\frac{1}{Y_f(1-\nu_f)}\left(1-\frac{\nu_f Y_f}{Y_s}\right) \tag{26}$$

V. CONCLUSION

So we prove that spectral density of Brownian (structural) fluctuations in coating is proportional to $\sim r_0^{-2}$.

The key question is the following: does the expression (23) is valid for arbitrary axial distribution of pressure or not? It seems that answer is 'yes'.

One can assume, for example, that mesa beam pressure distribution can be presented as sum (integral) of gaussian distributions. If this assumption is valid then one can easy scale the formula (26) for mesa beam by substitution

$$\int P_{\text{gauss}}^2(r)\, d\vec{r} \quad \Rightarrow \quad \int P_{\text{mesa beam}}^2(r)\, d\vec{r}$$

APPENDIX A: AUXILIARY INTEGRALS

Here we calculate auxiliary integrals:

$$\begin{aligned}
G_0 &= \frac{1}{\pi r_0^2}\int_{-\infty}^{\infty} e^{(x'^2+y'^2)/r_0^2}\,\frac{1}{r}\,dx'dy',\\
G_1 &= \frac{1}{\pi r_0^2}\int_{-\infty}^{\infty} e^{(x'^2+y'^2)/r_0^2}\,\frac{z}{r^3}\,dx'dy',\\
G_2 &= \frac{1}{\pi r_0^2}\int_{-\infty}^{\infty} e^{(x'^2+y'^2)/r_0^2}\,\frac{z^3}{r^5}\,dx'dy',\\
r &= \sqrt{(x-x')^2+(y-y')^2+z^2}
\end{aligned}$$

Integral G_0

$$\begin{aligned}
G_0 &= \int_{-\infty}^{\infty} dx'dy' \int_{-\infty}^{\infty} \frac{dk_x\, dk_y}{(2\pi)^2}\, e^{-(k_x^2+k_y^2)r_0^2/4}\, e^{ik_x(x'-x)+ik_y(y'-y)} e^{ik_x x+ik_y y}\,\frac{1}{r} =\\
&= \int_{-\infty}^{\infty} \frac{dk_x dk_y}{(2\pi)^2}\, e^{-(k_x^2+k_y^2)r_0^2/4}\, e^{ik_x x+ik_y y} \times G\\
G &= \int_{-\infty}^{\infty} dx'dy' e^{ik_x(x'-x)+ik_y(y'-y)}\,\frac{1}{\sqrt{((x-x')^2+(y-y')^2+z^2}}
\end{aligned}$$

We can take integral G over $dx'dy'$ using notation $k_\perp = \sqrt{k_y^2+k_z^2}$ (see also formula 2.5.24.1 in [8] and formula 2.12.4.28 in [9])

$$\begin{aligned}
G &= \int_{-\infty}^{\infty} dx'dy' e^{ik_x(x'-x)+ik_y(y'-y)}\,\frac{1}{\sqrt{(x-x')^2+(y-y')^2+z^2}} =\\
&= \int_0^\infty r'dr' \int_0^{2\pi} d\phi e^{ik_\perp r'\cos\phi)}\,\frac{1}{\sqrt{z^2+r'^2}} = \int_0^\infty r'dr' 2\pi \frac{J_0(k_\perp r')}{\sqrt{z^2+r'^2}} = \frac{2\pi}{k_\perp} e^{-k_\perp z},
\end{aligned}$$

where $J_0(x)$ is Bessel function of zero order.

Substituting G into (A1) we have:

$$\begin{aligned}
G_0 &= \int_{-\infty}^{\infty} \frac{dk_x dk_y}{(2\pi)^2}\, e^{-(k_x^2+k_y^2)r_0^2/4}\, e^{ik_x x+ik_y y} \times \frac{2\pi}{k_\perp} e^{-k_\perp z} = \int_0^\infty \int_0^{2\pi} \frac{k_\perp\, dk_\perp\, d\phi}{2\pi}\, e^{-k_\perp^2 r_0^2/4}\, e^{ik_\perp r\cos\phi} \times \frac{1}{k_\perp} e^{-k_\perp z} =\\
&= \int_0^\infty dk_\perp\, e^{-k_\perp^2 r_0^2/4}\, J_0(k_\perp r) \times e^{-k_\perp z}, \quad \text{below we use formula 2.12.9 from [9]:}\\
G_0|_{z=0} &= \frac{\sqrt{\pi}}{r_0}\exp\left(\frac{-r^2}{2r_0^2}\right) I_0\left(\frac{-r^2}{2r_0^2}\right), \quad r = \sqrt{x^2+y^2},
\end{aligned} \tag{A1}$$

where $I_0(x)$ is modified Bessel function of zero order.

Integral G_1 Using formula

$$\frac{z}{r^3} = -\partial_z\left(\frac{1}{r}\right)$$

we have:

$$\begin{aligned} G_1 &= \int_{-\infty}^{\infty} dx'dy' \int_{-\infty}^{\infty} \frac{dk_x\, dk_y}{(2\pi)^2}\, e^{-(k_x^2+k_y^2)r_0^2/4}\, e^{ik_x(x'-x)+ik_y(y'-y)} e^{ik_x x+ik_y y} \times \partial_z\left(\frac{-1}{r}\right) = \\ &= \int_{-\infty}^{\infty} \frac{dk_x dk_y}{(2\pi)^2}\, e^{-(k_x^2+k_y^2)r_0^2/4}\, e^{ik_x x+ik_y y} \times (-1)\frac{\partial}{\partial z}\, G \end{aligned}$$

Finally we obtain

$$G_1 = \int_{-\infty}^{\infty} \frac{dk_x dk_y}{2\pi}\, e^{-(k_x^2+k_y^2)r_0^2/4}\, e^{ik_x x+ik_y y}\, e^{-k_\perp z}, \quad G_1|_{z=0} = \frac{2}{r_0^2}\, e^{(x^2+y^2)/r_0^2}. \tag{A2}$$

Integral G_2. Using formula:

$$\frac{z^3}{r^5} = \frac{1}{3}\left(\frac{z\,\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\right)\frac{1}{r}$$

one can obtain

$$G_2 = \int_{-\infty}^{\infty} \frac{dk_x dk_y}{2\pi}\, e^{-(k_x^2+k_y^2)r_0^2/4}\, e^{ik_x x+ik_y y}\, \frac{1}{3k_\perp}\left(\frac{z\,\partial^2}{\partial z^2} - \frac{\partial}{\partial z}\right) e^{-k_\perp z} = \tag{A3}$$

$$= \frac{1}{3}\int_{-\infty}^{\infty} \frac{dk_x dk_y}{2\pi}\, e^{-(k_x^2+k_y^2)r_0^2/4}\, e^{ik_x x+ik_y y}\left(k_\perp z + 1\right) e^{-k_\perp z}, \tag{A4}$$

$$G_2|_{z=0} = \frac{2}{3r_0^2}\, e^{(x^2+y^2)/r_0^2}. \tag{A5}$$

APPENDIX B: CALCULATION OF u_x, u_y, u_z,

Calculation of u_z. Using expression (A2) for auxiary integral G_1 one can find that contribution of second term in figure brackets in (10) is zero in limit $z \to 0$. So we calculate the contribution of first term in figure brackets in (10) using expression for G_0 (A1):

$$u_z(x,y,z) = \frac{(1-\nu_s^2)F_0}{\pi E_s} \times \frac{\sqrt{\pi}}{r_0}\, \exp\left(\frac{-r^2}{2r_0^2}\right) I_0\left(\frac{-r^2}{2r_0^2}\right), \quad r = \sqrt{x^2+y^2} \tag{B1}$$

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