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Notes on Relating Sine-Gaussian Waveforms to Astrophysical Source Strengths

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Partly to provide an independent cross check of a small part of the GRB-triggered analysis, and mostly just to educate myself as a reviewer of that analysis, I've put together the following notes on relating sine-Gaussian waveforms to astrophysical source strengths. (To make it easier for others to check these calculations, some details of the integral formulae used are included here.)

Within the burst working group, it is customary to quantify analysis sensitivity in terms of $h_{\rm rss}$ (root sum square strain), which is defined generically by[1]

$$h_{\rm rss} \quad = \quad \sqrt{\int_{-\infty}^{+\infty} |h|^2 dt} \qquad (1)$$

The sine-Gaussian waveform is defined by[1]

$$h_{\rm SG}(t+t_0) \quad = \quad h_0 \ \sin(2\pi f_0 t)\, e^{-t^2/\tau^2} \qquad (2)$$

where it's convenient to define a parameter $Q = \sqrt{2}\pi\tau f_0$. The Fourier tranform of $h_{\rm SG}$ is Gaussian and centered on $f = f_0$ with standard deviation $\sigma_f = f_0/Q$.

It is straightforward to compute $h_{\rm rss}$ for the sine-Gaussian (define $t_0 = 0$ for simplicity):

$$h_{\rm rss}^2 \quad = \quad h_0^2 \int_{-\infty}^{+\infty} \sin^2(2\pi f_0 t)\, e^{-2t^2/\tau^2} dt \qquad (3)$$

$$= \quad \frac{1}{2}h_0^2 \int_{-\infty}^{+\infty} [1 - \cos(4\pi f_0 t)] e^{-2t^2/\tau^2} dt \tag{4}$$

where one can use Gradshteyn & Ryzhik[2] formulae 3.321.3 and 3.896.4:

$$\int_0^{+\infty} e^{-q^2x^2} dx \quad = \quad \frac{\sqrt{\pi}}{2q} \tag{5}$$

$$\int_0^{+\infty} e^{-\beta x^2} \cos(bx) dx \quad = \quad \frac{1}{2}\sqrt{\pi/\beta} e^{-b^2/(4\beta)} \tag{6}$$

to obtain

$$h_{\rm rss}^2 \quad = \quad h_0^2 \left[\frac{Q}{4\sqrt{\pi} f_0}(1 - e^{-Q^2})\right] \tag{7}$$

where the last factor in parentheses can be accurately approximated as unity for $Q >\sim 2$. Define a "High-Q" $h_{\rm rss}^{\rm HQ}$ by the dominant term:

$$(h_{\rm rss}^{\rm HQ})^2 \quad \equiv \quad h_0^2 \left[\frac{Q}{4\sqrt{\pi} f_0}\right] \tag{8}$$

Now let's address energy associated with this waveform. From Shapiro & Teukolsky[3] equation 16.1.12 (see also Misner, Thorne & Wheeler[4], equation 37.30), the instantaneous energy flux associated with a strain waveform propagating along the z direction is given by

$$T^{0z} \quad = \quad \frac{1}{16\pi}\frac{c^3}{G} < (\dot{h}_+)^2 + (\dot{h}_\times)^2 > \tag{9}$$

where the angle brackets denote a spatial average over several wavelengths.

In the following, I will compute the total wave energy under the following (admittedly artificial) assumptions:

- The gravitational wave is polarized wit $h_+(t) = h_{\rm SG}(t)$
- The gravitational wave emission is isotropic in its magnitude
- The wave energy passing through a small area can be computed from $\int(\dot{h})^2 dt$ without local spatial averaging

With these assumptions, the wave energy passing through a sphere of radius r centered upon the source is

$$E_{\rm GW} \quad = \quad (4\pi r^2) \times \frac{1}{16\pi}\frac{c^3}{G}\int_{-\infty}^{\infty} (\dot{h}_{\rm SG})^2 dt \tag{10}$$

Given the nearly sinusoidal (central frequency f_0) behavior of the sine-Gaussian waveform for high-Q, a quick 'n dirty estimate of the wave energy is

$$E_{\rm GW} = (4\pi r^2)\frac{1}{16\pi}\frac{c^3}{G}(2\pi f_0)^2\int_{-\infty}^{+\infty}(h_{\rm SG})^2 dt \quad (11)$$

$$= \frac{r^2}{4}\frac{c^3}{G}\,(2\pi f_0)^2\,(h_{\rm rss}^{\rm HQ})^2 \quad (12)$$

Plugging in some numbers, the quick 'n dirty expression becomes

$$E_{\rm GW} = (3.8\times 10^{47} J)\left(\frac{r}{100{\rm Mpc}}\right)^2\left(\frac{f_0}{100Hz}\right)^2\left(\frac{h_{\rm rss}^{\rm HQ}}{10^{-21}{\rm Hz}^{-1/2}}\right)^2 \quad (13)$$

$$= (2.1 M_{\rm sun}c^2)\left(\frac{r}{100{\rm Mpc}}\right)^2\left(\frac{f_0}{100Hz}\right)^2\left(\frac{h_{\rm rss}^{\rm HQ}}{10^{-21}{\rm Hz}^{-1/2}}\right)^2 \quad (14)$$

For reference, an upper limit on $h_{\rm rss}^{\rm HQ}$ of 6×10^{-21} $\rm Hz^{-1/2}$ at $f_0 = 250$ Hz for a source at 800 Mpc gives an upper limit on $E_{\rm GW}$ of about 3×10^4 solar masses. If one assumes that there is comparable energy in $h_\times$ polarization, then these energy values must be roughly doubled for a given $h_{\rm rss}$ sensitivity.

Just to be on the safe side in regard to low-Q waveforms, let's evaluate the integral $I_{\dot h} \equiv \int(\dot h_{\rm SG})^2 dt$ explicitly:

$$I_{\dot h} = \int_{-\infty}^{+\infty}\left[\frac{d}{dt}\left(h_0\,\sin(2\pi f_0 t)e^{-t^2/\tau^2}\right)\right]^2 dt \quad (15)$$

$$= h_0^2\int_{-\infty}^{+\infty} e^{-2t^2/\tau^2}\left[(2\pi f_0)\cos(2\pi f_0 t) - (\frac{2t}{\tau^2})\sin(2\pi f_0 t)\right]^2 dt \quad (16)$$

$$= h_0^2\Big\{4\pi^2 f_0^2\int_{-\infty}^{+\infty} e^{-2t^2/\tau^2}\cos^2(2\pi f_0 t)dt \quad (17)$$

$$-\frac{8\pi f_0}{\tau^2}\int_{-\infty}^{+\infty} e^{-2t^2/\tau^2} t\cos(2\pi f_0 t)\sin(2\pi f_0 t)dt \quad (18)$$

$$+\frac{4}{\tau^4}\int_{-\infty}^{+\infty} e^{-2t^2/\tau^2} t^2\sin^2(2\pi f_0 t)dt\Big\} \quad (19)$$

$$= h_0^2\Big\{4\pi^2 f_0^2\int_{-\infty}^{+\infty} e^{-2t^2/\tau^2}\frac{1}{2}(1+\cos(4\pi f_0 t))dt \quad (20)$$

$$-\frac{8\pi f_0}{\tau^2}\int_{-\infty}^{+\infty} e^{-2t^2/\tau^2} t\frac{1}{2}\sin(4\pi f_0 t)dt \quad (21)$$

$$+\frac{4}{\tau^4}\int_{-\infty}^{+\infty} e^{-2t^2/\tau^2} t^2\frac{1}{2}(1-\cos(4\pi f_0 t)dt\Big\} \quad (22)$$

where the first integral can be evaluated from the formulae used above for evaluating $h_{\rm rss}^2$, and the second and third integrals can be evaluated from Gradshteyn & Ryzhik[2] formulae 3.952.1, 3.461.2, and 3.952.4:

$$\int_0^{+\infty} x e^{-p^2x^2}\sin(ax)dx = \frac{a\sqrt{\pi}}{4p^3}\,e^{-a^2/(4p^2)} \tag{23}$$

$$\int_0^{+\infty} x^{2n} e^{-px^2}dx = \frac{(2n-1)!!}{2(2p)^n}\frac{\sqrt{\pi}}{\sqrt{p}} \tag{24}$$

$$\int_0^{+\infty} x^2 e^{-p^2x^2}\cos(ax)dx = \sqrt{\pi}\frac{(2p^2-a^2)}{8p^5}\,e^{-a^2/(4p^2)} \tag{25}$$

(Note that the parameter p has differing meanings in the above expressions taken directly from Gradshteyn & Ryzhik.)

One obtains:

$$I_{\dot{h}} = h_0^2\Big\{4\pi^2 f_0^2\left[\frac{\sqrt{\pi}}{2\sqrt{2}}\,\tau\,(1+e^{-Q^2})\right] \tag{26}$$

$$-\frac{8\pi f_0}{\tau^2}\left[\frac{\pi^{3/2} f_0\tau^3}{2\sqrt{2}}\,e^{-Q^2}\right] \tag{27}$$

$$+\frac{4}{\tau^4}\left[\frac{\sqrt{\pi}\tau^3}{8\sqrt{2}}[1-(1-2Q^2)\,e^{-Q^2}]\right]\Big\} \tag{28}$$

$$= (2\pi f_0)^2\,(h_{\rm rss}^{\rm HQ})^2\times[1+\frac{1}{2Q^2}(1-e^{-Q^2})] \tag{29}$$

The correction factor in square brackets, which approaches $3/2$ as $Q\to 0$ and 1 as $Q\to\infty$, is plotted in figure 1 for Q values ranging from 0 to 10 and is important for Q values less than ~ 2. The wave energy expression in equation 14 is affected proportionately.

References

[1] B. Abbot *et al.*, "First Upper Limits from LIGO on Gravitational Wave Bursts", to appear in Phys. Rev. D, gr-qc/0312056 (December 2003).

[2] I.S. Gradshteyn and I.M. Ryzhik, "Table of Integrals, Series, and Products", Academic Press, 1980.

[3] S.L. Shapiro and S.A. Teukolsky, "Black Holes, White Dwarfs, and Neutron Stars", John Wiley & Sons, Inc., 1983.

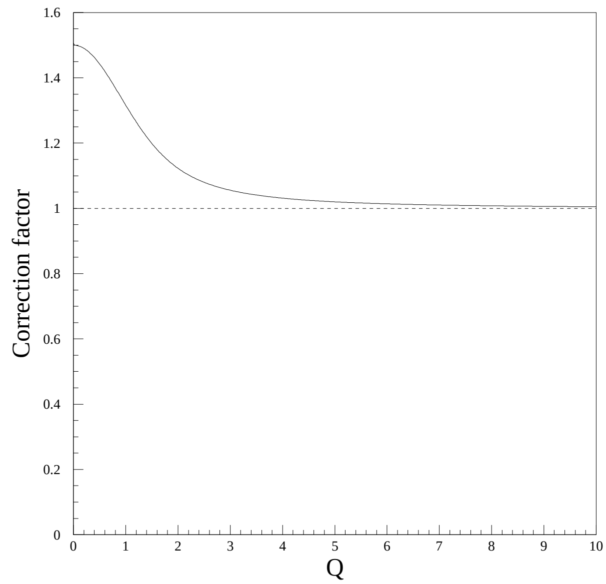

Figure 1: Energy correction factor *vs.* Q

[4] C.W. Misner, K.S. Thorne, and J.A. Wheeler, "Gravitation", W.H. Freeman & Co., 1973.