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# Encorporating coating anisotropy into coating thermal noise

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Start with a description of an anisotropic multilayer coating from Hoff [1]:

$$\begin{aligned}\epsilon_{rr} &= \sigma_{rr}/Y_{||} - \sigma_{||}\sigma_{\theta\theta}/Y_{||} - \sigma_{\perp}\sigma_{zz}/Y_{\perp} \\ \epsilon_{\theta\theta} &= -\sigma_{||}\sigma_{rr}/Y_{||} + \sigma_{\theta\theta}/Y_{||} - \sigma_{\perp}\sigma_{zz}/Y_{\perp} \\ \epsilon_{zz} &= -\sigma_{\perp}\sigma_{rr}/Y_{\perp} - \sigma_{\perp}\sigma_{\theta\theta}/Y_{\perp} + \sigma_{zz}/Y_{\perp}.\end{aligned} \tag{1}$$

Here, $Y_{||}$ is the Young's modulus for stresses causing strains entirely within the plane parallel to the coating layers. $Y_{\perp}$ is the Young'g modulus for stresses causing strains perpendicular to the coating layers. There are two Poisson's ratios, $\sigma_{||}$ for stresses and strains both with the plane parallel to the coating layers, and $\sigma_{\perp}$ for when either the stress or the stain is perpendicular to the coating layers.

The matrix in Eq. 1 can be inverted, to read

$$\begin{aligned}\sigma_{rr} &= (\lambda_1 + 2\mu_1)\,\epsilon_{rr} + \lambda_1\epsilon_{\theta\theta} + \lambda_2\epsilon_{zz} \\ \sigma_{\theta\theta} &= \lambda_1\epsilon_{rr} + (\lambda_1 + 2\mu_1)\,\epsilon_{\theta\theta} + \lambda_2\epsilon_{zz} \\ \sigma_{zz} &= \lambda_2\epsilon_{rr} + \lambda_2\epsilon_{\theta\theta} + (\lambda_2 + 2\mu_2)\,\epsilon_{zz},\end{aligned} \tag{2}$$

where

$$\lambda_1 = -Y_{||}\left(\sigma_{\perp}^2 Y_{||} + \sigma_{||}Y_{\perp}\right)/\left(\left(\sigma_{||}+1\right)\left(2\sigma_{\perp}^2 Y_{||} - \left(1-\sigma_{||}\right)Y_{\perp}\right)\right), \tag{3}$$

$$\mu_1 = Y_{||}/\left(2\left(1+\sigma_{||}\right)\right), \tag{4}$$

$$\lambda_2 = \sigma_{\perp}Y_{||}Y_{\perp}/\left(-2\sigma_{\perp}^2 Y_{||} + \left(1-\sigma_{||}\right)Y_{\perp}\right), \tag{5}$$

$$\mu_2 = Y_{\perp}\left(\sigma_{\perp}Y_{||} - \left(1-\sigma_{||}\right)Y_{\perp}\right)/\left(2\left(2\sigma_{\perp}^2 2Y_{||} - \left(1-\sigma_{||}\right)Y_{\perp}\right)\right). \tag{6}$$

Equation 2 is the anisotropic equivalent of Gretarsson's equation A2 in ref. [2]. Following the same procedure as in that reference, the equivalent of Gretarsson's equation A4 is found;

$$\begin{aligned}\epsilon'_{rr} &= \epsilon_{rr} \\ \epsilon'_{\theta\theta} &= \epsilon_{\theta\theta} \\ \epsilon'_{zz} &= (\lambda - \lambda_2)/(\lambda_2 + 2\mu_2)(\epsilon_{rr} + \epsilon_{\theta\theta}) + (\lambda + 2\mu)/(\lambda_2 + 2\mu_2)\,\epsilon_{zz} \\ \epsilon'_{rz} &= \epsilon_{rz} \\ \sigma'_{rr} &= (\lambda_1 + 2\mu_1)\,\epsilon_{rr} + \lambda_1\epsilon_{\theta\theta} + \lambda_2\epsilon'_{zz} \\ \sigma'_{\theta\theta} &= \lambda_1\epsilon_{rr} + (\lambda_1 + 2\mu_2)\,\epsilon_{\theta\theta} + \lambda_2\epsilon'_{zz} \\ \sigma'_{zz} &= \sigma_{zz} \\ \sigma'_{rz} &= \sigma_{rz}.\end{aligned} \tag{7}$$

Here, all the parameters without numerical subscrips refer to values in the substrate.

Following the method of Gretarsson as in [2] with the changes noted above for an anisotropic modulus, the effective loss angle $\phi_{\rm readout}$ for thermal noise calculations in an interferometer can be found.

$$\phi_{\rm readout} = \phi_{\rm substrate} + d/\left(\sqrt{\pi}wY_{\perp}\right)$$

$$
\begin{aligned}
&\left(\left(Y/\left(1-\sigma_{\perp}^{2}\right)-2\sigma_{\perp}^{2}YY_{||}/\left(Y_{\perp}\left(1-\sigma^{2}\right)\left(1-\sigma_{||}\right)\right)\right)\phi_{\perp}\\
&+Y_{||}\sigma_{\perp}\left(1-2\sigma\right)/\left(\left(1-\sigma_{||}\right)\left(1-\sigma\right)\right)\left(\phi_{||}-\phi_{\perp}\right)\\
&+Y_{||}Y_{\perp}\left(1+\sigma\right)\left(1-2\sigma\right)^{2}/\left(Y\left(1-\sigma_{||}^{2}\right)\left(1-\sigma\right)\right)\phi_{||}\right).
\end{aligned} \tag{8}
$$

This is to be compared to equation 20 in reference [2].

The limit of Eq. 8 where all the Poisson ratios are small, $\sigma_{||}=\sigma_{\perp}=\sigma=0$, gives

$$\phi_{\text{readout}}=\phi_{\text{substrate}}+d/\left(\sqrt{\pi}w\right)\left(Y/Y_{\perp}\phi_{\perp}+Y_{||}/Y\phi_{||}\right). \tag{9}$$

The limit when $Y_{||}=Y_{\perp}=Y'$ and $\sigma_{||}=\sigma_{\perp}=\sigma'$, ie when the coating is assumed isotropic except in its loss angles, Eq. 8 reproduces Eq. 21 in reference [2].

The values for $Y_{\perp}, Y_{||}, \phi_{\perp}$ and $\phi_{||}$ can be calculated from the values of the isotropic materials that make up the layers of the coating. For a coating of two layers,

$$Y_{\perp}=\left(d_1+d_2\right)/\left(d_1/Y_1+d_2/Y_2\right), \tag{10}$$

$$Y_{||}=\left(Y_1d_1+Y_2d_2\right)/\left(d_1+d_2\right), \tag{11}$$

$$\phi_{\perp}=Y_{\perp}\left(\phi_1d_1/Y_1+\phi_2d_2/Y_2\right)/\left(d_1+d_2\right), \tag{12}$$

$$\phi_{||}=\left(Y_1\phi_1d_1+Y_2\phi_2d_2\right)/\left[Y_{||}\left(d_1+d_2\right)\right], \tag{13}$$

where $d_1$ and $d_2$ are the total thicknesses, $Y_1$ and $Y_2$ are the isotropic Young's moduli, and $\phi_1$ and $\phi_2$ are the isotropic loss angles of the two materials that make up the coating.

To use the full formula in Eq. 8, values for the Poisson ratios are needed. The Poisson ratio connecting stress in the $z$ direction with strains in the plane of the coating, $\theta-r$, $\sigma_{\perp}$ can be found from

$$\sigma_{\perp}=\left(\sigma_1Y_1d_1+\sigma_2Y_2d_2\right)/\left(Y_1d_1+Y_2d_2\right). \tag{14}$$

The equivalent formula for $\sigma_{||}$ is more complicated. The value for $\sigma_{||}$ can be found by numerically solving

$$\lambda_1=\left(\sigma_1Y_1d_1/\left(\left(1+\sigma_1\right)\left(1-2\sigma_1\right)\right)+\sigma_2Y_2d_2/\left(\left(1+\sigma_2\right)\left(1-2\sigma_2\right)\right)\right)/\left(d_1+d_2\right). \tag{15}$$

For $SiO_2/Ta_2O_5$ coatings in the appropriate ratios to make an HR coating for 1.062 $\mu$m light, these values are

$$\sigma_{\perp} = 0.19, \tag{16}$$

$$\sigma_{||} = 0.21. \tag{17}$$

# References


[1] N. J. Hoff in *Engineering Laminates*, ed. A. G. H. Hoff (John Wiley and Sons, New York, 1949).

[2] G. M. Harry, A. M. Gretarsson, P. R. Saulson, S. E. Kittleberger, S. D. Penn, W. J. Startin, S. Rowan, M. M. Fejer, D. R. M. Crooks, G. Cagnoli, J. Hough, and N. Nakagawa, *Class. Quantum Grav.***19***(2003)897.*