LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY

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Optical Coupling of Vertical Displacement to Cavity Length

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1 Introduction

Calculations of core optic vertical displacement to cavity length coupling, due solely to optical geometry, is documented in this memorandum for both initial LIGO and advanced LIGO.

The optical layout designs[1] give the orientation of the core optics in terms of global coordinates. These orientations are transformed into local coordinates[2] for each building at each of the two LIGO observatories. The effect of vertical displacement, parallel to the local gravity vector, is then calculated for the power recycling cavity (PRC), signal recycling cavity (SRC) and the Fabry-Perot (FP) arm cavities.

The coupling for a given degree of freedom needs to be applied to the noise requirements for that d.o.f. Displacement noise requirements for auxiliary optics[3] are defined in T010097.

An alternative advanced LIGO layout with horizontally wedged optics is also being considered. Such a design would essentially eliminate vertical to length coupling from optical geometry. Note that there are other potential sources of vertical-length coupling such as mechanical coupling in the suspension or inadvertent electronic sensing feedback coupling.

2 Initial LIGO

Positions and orientations of the Core Optics are taken from D. Rose's "IFO COC and Beam Coordinate Data", E990083-A; except note that Rose has listed the negative of the surface normal vectors defined in D. Coyne's "Determination of the Wedge Angles for the Core Optics Components", T970091-00 (defined in figures 7 and 8); The definition in T970091 is *generally* the outward surface normal, but not in all cases; Here the outward surface normal is used in *all* cases.

2.1 2km Interferometer (LHO)

The coupling factors for vertical displacement of each of the core optics of the 2 km long interferometer at the LIGO Hanford Observatory (LHO) are listed in Table 1. In all of the following tables, alpha is the wedge angle, and beta is the orientation angle of the high reflectance surface normal vector to horizontal. The notation betaG refers to the optic orientation in the global coordinate system (i.e. defined by the plane of the interferometer), whereas betaL refers to the optic orientation in the local coordinate system (i.e. defined relative to the local gravity vector). The coupling factors are defined for the common (+) and differential (-) lengths of each of the cavities. The sign notation for the alpha and beta angles are defined in the derivation sections which follow.

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[1] The optical layout for initial LIGO is defined in:
D. Rose, IFO COC and Beam Coordinate Data, E990083-A
D. Coyne, Determination of the Wedge Angles for the Core Optics Components, T970091
For advanced LIGO the optical layout is defined in:
D. Coyne, Optical Layout for Advanced LIGO, T010076-01.

[2] using Tables 10-14 and Tables 25-27 of:
A. Lazzarini, "Determination of Local and Global Coordinate Axes for the LIGO Sites", T980044-10

[3] P. Fritschel, Auxiliary Suspended Optics Displacement Noise Requirements, T010097-00.
N.B.: Note that at the time, T010097 wasn't sophisticated enough to consider the beamsplitter's vertical motion coupling to the different d.o.f. and should be revised based on the coupling factors in this memorandum.

Table 1. Vertical-Length Coupling Factors for the initial LIGO 2km Interferometer

	alpha	betaG	betaL	PRC+	PRC-	FP+	FP-
RM HR (p3)	-4.19×10^{-2}	-1.89×10^{-2}	-1.83×10^{-2}	1.83×10^{-2}	0	0	0
FMy (p6)	0	3.32×10^{-3}	2.89×10^{-3}	-1.45×10^{-3}	-2.89×10^{-3}	0	0
ITMy HR (p8)	9.89×10^{-3}	0.	1.25×10^{-5}	-2.22×10^{-3}	-4.45×10^{-3}	-6.24×10^{-6}	-1.25×10^{-5}
BS bs (p4)	1.75×10^{-2}	6.9×10^{-3}	6.45×10^{-3}	-9.44×10^{-3}	-6.22×10^{-4}	0	0
FMx (p9)	0	3.31×10^{-3}	2.88×10^{-3}	-1.44×10^{-3}	-2.88×10^{-3}	0	0
ITMx HR (p11)	9.89×10^{-3}	0.	-6.19×10^{-4}	-2.22×10^{-3}	-4.45×10^{-3}	3.1×10^{-4}	6.19×10^{-4}
ETMx HR	0	0.	3.06×10^{-4}	0	0	-1.53×10^{-4}	-3.06×10^{-4}
ETMy HR	0	0.	-3.26×10^{-4}	0	0	1.63×10^{-4}	3.26×10^{-4}

2.2 4km Interferometer (LHO)

Table 1. Vertical-Length Coupling Factors for the initial LIGO 4km Interferometer at LHO

	alpha	betaG	betaL	PRC+	PRC-	FP+	FP-
RM HR (-p3)	-4.19×10^{-2}	-1.89×10^{-2}	-1.95×10^{-2}	1.95×10^{-2}	0	0	0
BS bs (pc)	1.75×10^{-2}	-6.88×10^{-3}	-7.33×10^{-3}	3.11×10^{-4}	-2.01×10^{-2}	0	0
ITMx HR (p7)	2.04×10^{-2}	0.	-6.19×10^{-4}	-4.58×10^{-3}	-9.16×10^{-3}	3.1×10^{-4}	6.19×10^{-4}
ITMy HR (p9)	2.04×10^{-2}	0.	1.25×10^{-5}	-4.58×10^{-3}	-9.16×10^{-3}	-6.24×10^{-6}	-1.25×10^{-5}
ETMx HR	0	0.	-7.84×10^{-6}	0	0	3.92×10^{-6}	7.84×10^{-6}
ETMy HR	0	0.	-6.39×10^{-4}	0	0	3.2×10^{-4}	6.39×10^{-4}

2.3 4km Interfeometer (LLO)

Table 1. Vertical-Length Coupling Factors for the initial LIGO 4km Interferometer at LLO

	alpha	betaG	betaL	PRC+	PRC-	FP+	FP-
RM HR (-p3)	-4.19×10^{-2}	-1.89×10^{-2}	-1.92×10^{-2}	1.92×10^{-2}	0	0	0
BS bs (pc)	1.75×10^{-2}	-6.88×10^{-3}	-6.67×10^{-3}	-1.54×10^{-4}	-1.92×10^{-2}	0	0
ITMx HR (p7)	2.04×10^{-2}	0.	-3.12×10^{-4}	-4.58×10^{-3}	-9.16×10^{-3}	1.56×10^{-4}	3.12×10^{-4}
ITMy HR (p9)	2.04×10^{-2}	0.	-6.11×10^{-4}	-4.58×10^{-3}	-9.16×10^{-3}	3.05×10^{-4}	6.11×10^{-4}
ETMx HR	0	0.	-3.15×10^{-4}	0	0	1.57×10^{-4}	3.15×10^{-4}
ETMy HR	0	0.	-1.88×10^{-5}	0	0	9.4×10^{-6}	1.88×10^{-5}

3 Advanced LIGO

The Compensation Plates (CP) are not yet included in the vertical-length coupling analysis. Since we are considering use of a CP surface for pick-off beams, significant wedge angles will be needed and the vertical-length coupling may be important.

3.1 4km Interferometer (LHO)

Table 1. Vertical-Length Coupling Factors for the advanced LIGO 4km Interferometer at LHO

	alpha	betaG	betaL	PRC+	PRC-	SRC+	SRC-	FP+	FP-
PRM	4.03×10^{-3}	1.81×10^{-3}	1.19×10^{-3}	-1.19×10^{-3}	0	0	0	0	0
SRM	-4.05×10^{-3}	1.82×10^{-3}	1.83×10^{-3}	0	0	-1.83×10^{-3}	0	0	0
BS	2.27×10^{-2}	9.25×10^{-3}	9.7×10^{-3}	-1.71×10^{-2}	-6.81×10^{-3}	-2.39×10^{-2}	-6.81×10^{-3}	0	0
ITMx	1.92×10^{-2}	0.	6.19×10^{-4}	-8.06×10^{-3}	-1.61×10^{-2}	-8.06×10^{-3}	-1.61×10^{-2}	-3.1×10^{-4}	-6.19×10^{-4}
ITMy	1.92×10^{-2}	0.	-1.25×10^{-5}	-7.27×10^{-3}	-1.45×10^{-2}	-7.27×10^{-3}	-1.45×10^{-2}	6.24×10^{-6}	1.25×10^{-5}
ETMx	0	0.	-7.84×10^{-6}	0	0	0	0	3.92×10^{-6}	7.84×10^{-6}
ETMy	0	0.	-6.39×10^{-4}	0	0	0	0	3.2×10^{-4}	6.39×10^{-4}

3.2 4km Interferometer (LLO)

Table 1. Vertical-Length Coupling Factors for the advanced LIGO 4km Interferometer at LLO

	alpha	betaG	betaL	PRC+	PRC-	SRC+	SRC-	FP+	FP-
PRM	4.03×10^{-3}	1.81×10^{-3}	1.5×10^{-3}	-1.5×10^{-3}	0	0	0	0	0
SRM	-4.05×10^{-3}	1.82×10^{-3}	1.21×10^{-3}	0	0	-1.21×10^{-3}	0	0	0
BS	2.27×10^{-2}	9.25×10^{-3}	9.04×10^{-3}	-1.27×10^{-2}	1.07×10^{-4}	-1.61×10^{-2}	1.07×10^{-4}	0	0
ITMx	1.92×10^{-2}	0.	3.12×10^{-4}	-7.68×10^{-3}	-1.54×10^{-2}	-7.68×10^{-3}	-1.54×10^{-2}	-1.56×10^{-4}	-3.12×10^{-4}
ITMy	1.92×10^{-2}	0.	6.11×10^{-4}	-8.05×10^{-3}	-1.61×10^{-2}	-8.05×10^{-3}	-1.61×10^{-2}	-3.05×10^{-4}	-6.11×10^{-4}
ETMx	0	0.	-3.15×10^{-4}	0	0	0	0	1.57×10^{-4}	3.15×10^{-4}
ETMy	0	0.	-1.88×10^{-5}	0	0	0	0	9.4×10^{-6}	1.88×10^{-5}

3.3 Input Test Mass (ITM)

The input test mass separates the power and signal recycled cavities from the Fabry-Perot (FP) arm cavity. Vertical motion of the ITM will effect all three cavities. Consider an optic with a vertical wedge angle of α, as depicted in Figure 1. We denote the high reflectance (HR) side as 1 and the anti-reflection (AR) side as 2. We define a positive wedge angle as that which results in a thick end up. The orientation of surface 1 relative to the local gravity vector, $\vec{g}$, is the angle β_1, as indicated in Figure X. The sign of the orientation angle is defined as positive if the left facing surface normal vector has a negative vertical component (where the FP arm cavity is to the right). The parameters α and β_1 are sufficient to define the geometry; There is no component of the optical rays out of the vertical plane.

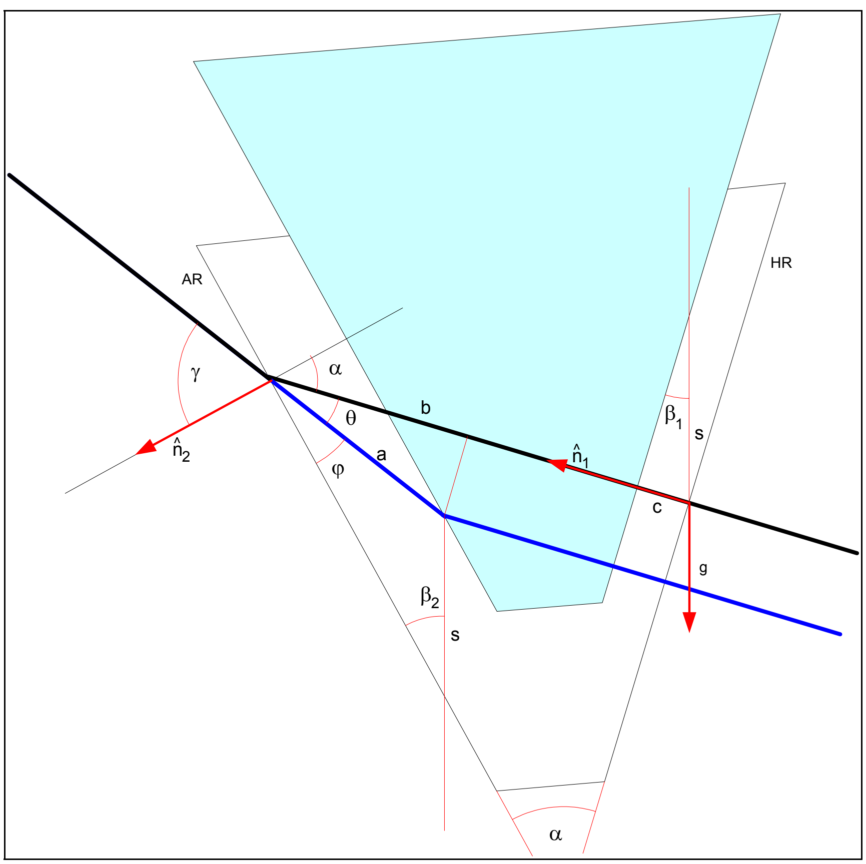

Figure 1. Vertical displacement of the ITM

If the ITM is displaced upward by a distance "s" (as depicted by the light blue ITM in Figure 1), then the optical path distance (OPD) on the AR side changes as follows:

$$\Delta_{AR} = a - n_o(b + c)$$

where n_o is the optic refraction index and the lengths a, b and c are defined in Figure 1.

By the law of sines:

$$\frac{a}{Sin\beta_2} = \frac{s}{Sin\varphi}$$

$$\frac{a}{s} = \frac{Sin(\beta_1 + \alpha)}{Sin(\frac{\pi}{2} - \gamma)} = \frac{Sin(\beta_1 + \alpha)}{Cos\gamma}$$

By the law of refraction:

$$n_o Sin\alpha = Sin\gamma$$

$$\frac{a}{s} = \frac{Sin(\beta_1 + \alpha)}{\sqrt{1 - n_o^2 Sin^2\alpha}}$$

Note that Sign(a) = Sign(β_1+α) as it should, so that the length a is added to Δ_{AR} when β_2 = (β_1+α) is positive and subtracted when β_2 is negative.

The length b is given as follows:

$$b = a\ Cos\theta = a\ Cos(\gamma - \alpha)$$

$$\frac{b}{s} = \frac{Sin(\beta_1 + \alpha)}{\sqrt{1 - n_o^2 Sin^2\alpha}}\left[Cos\alpha\sqrt{1 - n_o^2 Sin^2\alpha} + n_o Sin^2\alpha\right]$$

Note that Sign(b) = Sign(a), as it should, so that (n_o b) is subtracted from Δ_{AR} when β_2 = (β_1+α) is positive, according to the expression for Δ_{AR} above.

The length c is given as follows:

$$\frac{c}{s} = -Sin\beta_1$$

Note that Sign(c) = -Sign(β_1), as it should, so that c is subtracted from Δ_{AR} when β_1 is negative, according to the expression for Δ_{AR} above.

The OPD change on the AR side is then:

$$\frac{\Delta_{AR}}{s} = \frac{Sin(\beta_1 + \alpha)}{\sqrt{1 - n_o^2 Sin^2\alpha}}\left[1 - n_o Cos\alpha\sqrt{1 - n_o^2 Sin^2\alpha} - n_o^2 Sin^2\alpha\right] - n_o Sin\beta_1$$

This formula has been verified by ray trace analysis with the Mathematica™ package Optica™.

The effect of the AR side OPD change on the power recycling cavity lengths and on the signal recycling cavity lengths is as follows:

$$\frac{dl^{+}_{prc}}{ds} = \frac{dl^{+}_{src}}{ds} = \frac{\Delta_{AR}}{2s}$$

$$\frac{dl^{-}_{prc}}{ds} = \frac{dl^{-}_{src}}{ds} = \frac{\Delta_{AR}}{s}$$

The OPD change on the HR side is simply the distance c. The arm cavity common and differential length changes associated with vertical displacement of the ITM is thus:

$$\frac{dL^{+}}{ds} = \frac{\Delta_{HR}}{2s} = -\frac{Sin\beta_1}{2}$$

$$\frac{dL^{-}}{ds} = \frac{\Delta_{HR}}{s} = -Sin\beta_1$$

The formulation presented in Appendix 1 of T010076-01 takes only the horizontal component of the OPD change. Consider the above equation for $\frac{\Delta_{AR}}{s}$ when $\beta_1 = 0$ and using the small angle approximations, $Cos(\alpha) \approx 1$ and $Sin(\alpha) \approx \alpha$. The resulting approximation is:

$$\frac{\Delta_{AR}}{s} \approx \alpha\,(1 - n_o)$$

which is consistent with Appendix 1 of T010076-01. Similarly, one can show that for non-zero β_1, the small angle approximation is

$$\frac{\Delta_{AR}}{s} \approx (\alpha + \beta)\,(1 - n_o) - n_o\beta$$

For a reasonable range of wedge angles and orientation angles (±0.06 rad or ±3.4 deg), the error in the above equation is only ~10^{-4}, as shown in the plot below.

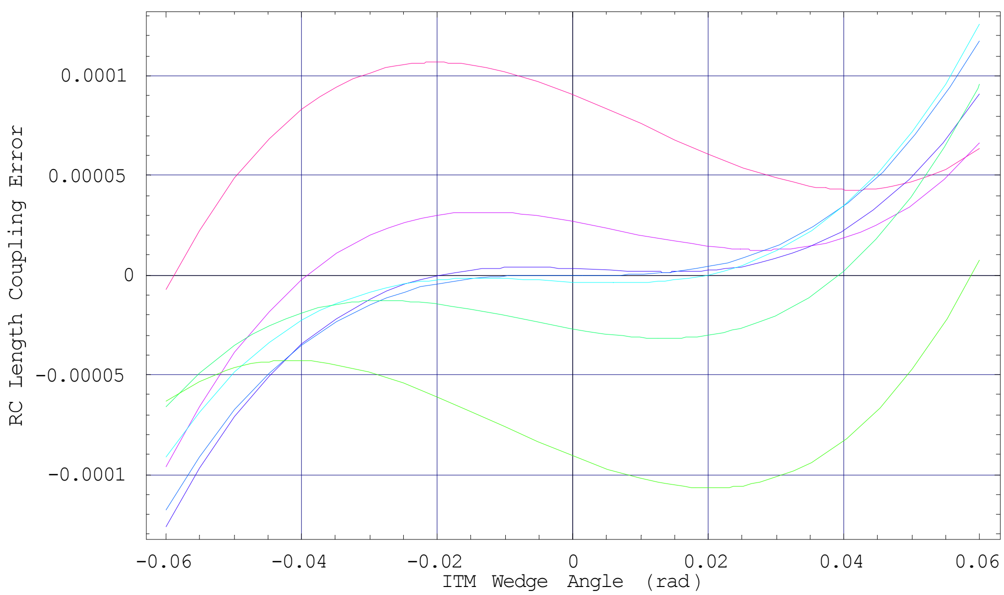

Figure 2. Approximation error (absolute) for the orientation angle, -0.06 < β_1 < 0.06 rad

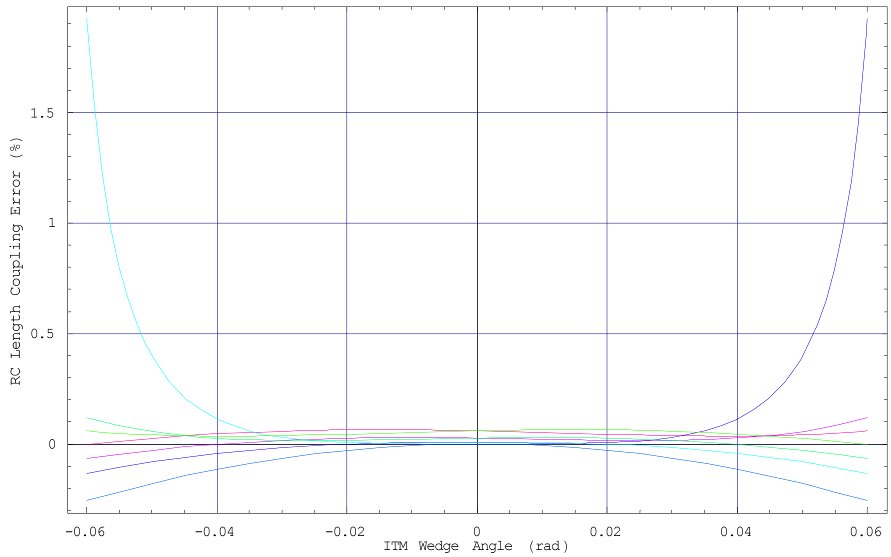

Figure 3. Approximation error (%) for the orientation angle, $-0.06 < \beta_1 < 0.06$ rad

3.4 End Test Mass (ETM)

Vertical motion of the ETM will affect the Fabry-Perot arm cavity length. The OPD change on the HR side is simply the distance c (defined in Figure 1). The arm cavity common and differential length changes associated with vertical displacement of the ETM is thus:

$$\frac{dL^+}{ds} = \frac{\Delta_{HR}}{2s} = -\frac{Sin\beta_1}{2}$$

$$\frac{dL^-}{ds} = \frac{\Delta_{HR}}{s} = -Sin\beta_1$$

3.5 Power Recycling Mirror (PRM)

Vertical motion of the PRM will affect the Power Recycling Cavity (PRC) length. The OPD change on the HR side is simply the distance c (defined in Figure 1). The PRC cavity common and differential length changes associated with vertical displacement of the PRM is thus:

$$\frac{dl^+_{prc}}{ds} = \frac{\Delta_{HR}}{s} = -Sin\beta_1$$

$$\frac{dl^-_{prc}}{ds} = 0$$

3.6 Signal Recycling Mirror (SRM)

Vertical motion of the SRM will affect the Signal Recycling Cavity (SRC) length. The OPD change on the HR side is simply the distance c (defined in Figure 1). The SRC cavity common and differential length changes associated with vertical displacement of the SRM is thus:

$$\frac{dl^{+}_{src}}{ds} = \frac{\Delta_{HR}}{s} = -Sin\beta_1$$

$$\frac{dl^{-}_{src}}{ds} = 0$$

3.7 Beamsplitter (BS)

Vertical motion of the BS will effect the lengths of both the power and signal recycling cavities. Consider a BS with a vertical wedge angle of α, as depicted in Figure 4. The notation for the surfaces of the PRC and SRC is per appendix 2 of T010076-01. We define a positive wedge angle, α, as that which results in a thick end up. The orientation of the splitting (50/50) surface relative to the local gravity vector, $\vec{g}$, is the angle β, as indicated in Figure 4. The sign of the orientation angle is defined as positive if the outward facing surface normal vector (for the splitting surface) has a positive vertical component. The parameters α and β are sufficient to define the geometry; There is no component of the optical rays out of the vertical plane.

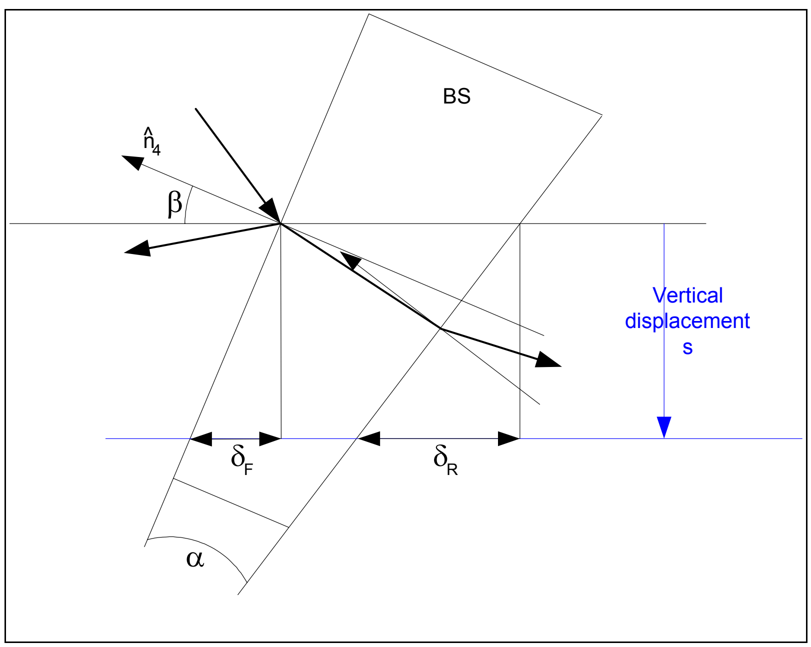

Figure 4. Elevation view of the BS, indicating vertical displacement

3.7.1 PRC

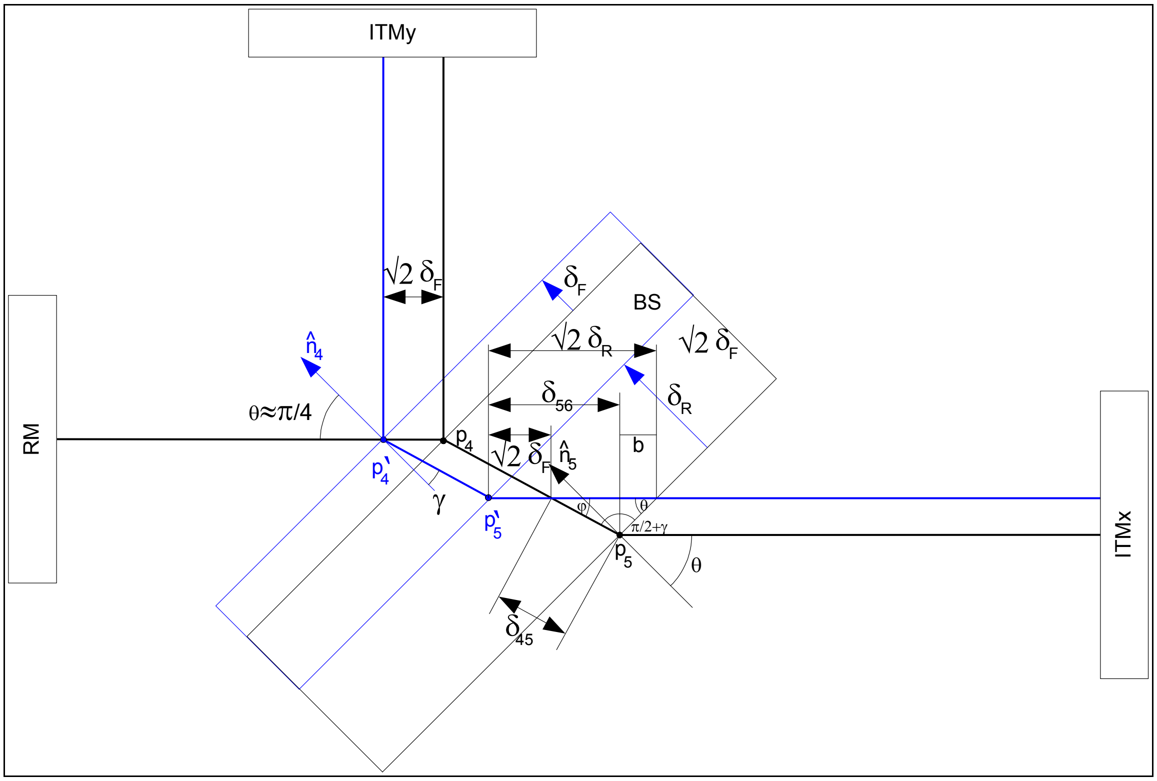

Figure 5. Power recycling cavity with vertically displaced BS

It can be shown that the power recycling cavity (PRC) differential length, l_m, change for a vertical displacement, s, of the BS is:

$$\frac{dl_m}{ds} = \sqrt{2}\tan(\beta+\alpha) - \frac{\tan(\beta+\alpha)-\tan(\beta)}{\sqrt{2}}\left(\sqrt{2n_o^2-1}+1\right)$$

The recycling cavity common length, l_p, change for a vertical displacement, s, of the BS is:

$$\frac{dl_p}{ds} = -\sqrt{2}\tan(\beta) + \frac{1}{2}\frac{dl_m}{ds}$$

It may appear that the power recycling cavity length sensitivity to BS vertical motion can be minimized by appropriate selection of the wedge angle, α, and the angle of incidence of the BS surface normal, β. The BS wedge angle was set to the maximum for manufacturability. The resulting separation of the BS 1st ghost beam at the ITM was just (barely) acceptable. In other words the BS wedge angle is constrained. It might seem that the inclination of the BS surface with respect to vertical could be varied in order to minimize either the RC common or differential lengths (l_p or l_m). However, given the RM and ITM vertical positions, there are no additional degrees of freedom (other than the FMs in the 2km IFO). In effect, the vertical inclination of the BS normal vector must be set so that the angular deviation in the reflected chief ray is (approximately) the same as the angular deviation of the transmitted ray. This is an approximation since the distances in the reflected and transmitted RC paths are different due to the Schnupp asymmetry.

For initial LIGO the BS wedge angle, $\alpha = 0.01745$ rad and the BS angle of inclination, $\beta = 0.00689$ rad. This results in power recycling cavity differential length coupling of $dl_m/ds \sim -1.5e\text{-}6$ and a power recycling cavity common length coupling of $dl_p/ds = -0.0097$ (as indicated in the figure below, where the points are from a ray trace analysis and the curves are the formulas above).

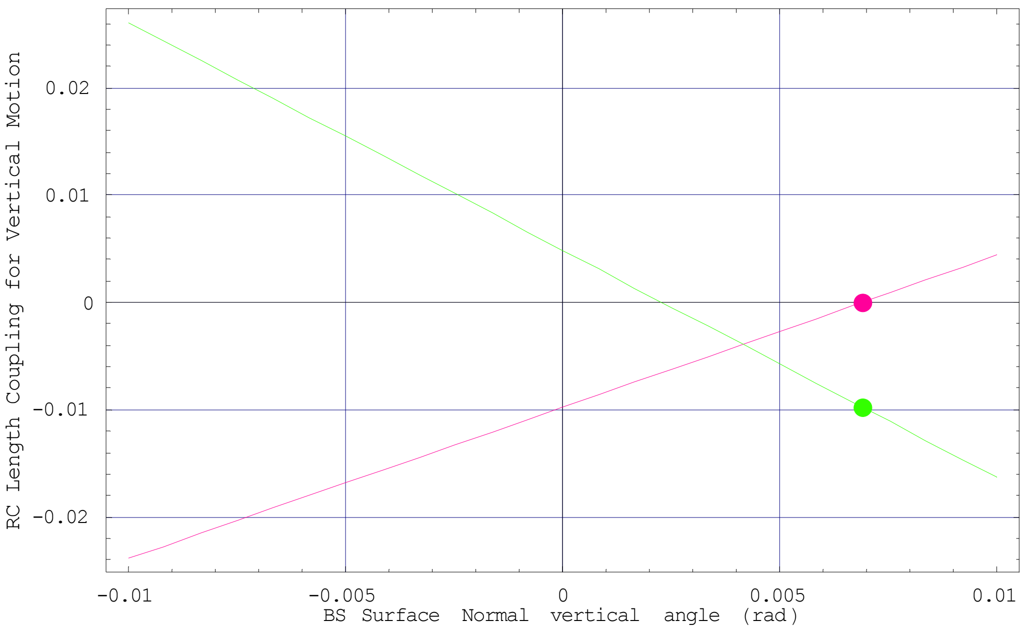

Figure 6. BS Vertical coupling to the PRC for the initial LIGO 4km interferometer
The lines are per the equations above and the points are from an Optica ray trace.

For advanced LIGO, the BS vertical motion will couple to common and differential length changes to both the power and signal recycling cavities. The BS wedge angle $\alpha = 0.02269$ rad and the BS angle of inclination $\beta = 0.00925$ rad[4]. Results from the formulas above and the optical ray trace analysis agree, as indicated in the figure below.

$$PRC\ common\ length\ coupling, \frac{dl_{p,p}}{ds} = -0.0129$$

$$PRC\ differential\ length\ coupling, \frac{dl_{p,m}}{ds} = 0.00042$$

$$SRC\ common\ length\ coupling, \frac{dl_{s,p}}{ds} = -0.0125$$

[4] The value $\beta = 0.00897$ rad reported in T010076-01 is incorrect. An error in the Mathematica notebook placed ITMy in an incorrect position (error on the order of 200 mm). Cavity lengths and Schnupp asymmetry is now correct.

$$SRC\ differential\ length\ coupling, \frac{dl_{s,m}}{ds} = 0.00042$$

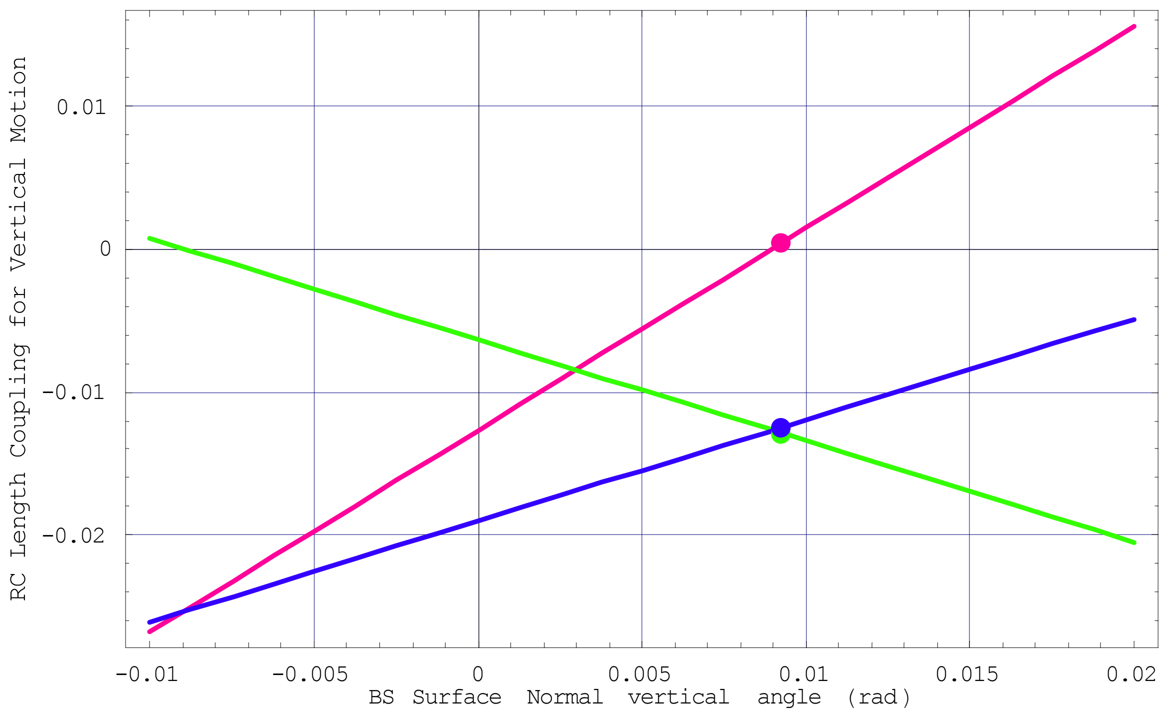

Figure 7: Advanced LIGO BS vertical displacement to PRC and SRC length
red line is PRC or SRC differential length change per the above equation
green line is PRC common length change per the above equation
blue line is SRC common length change per the above equation
dots are interpreted from an Optica ray trace analysis

3.7.2 SRC

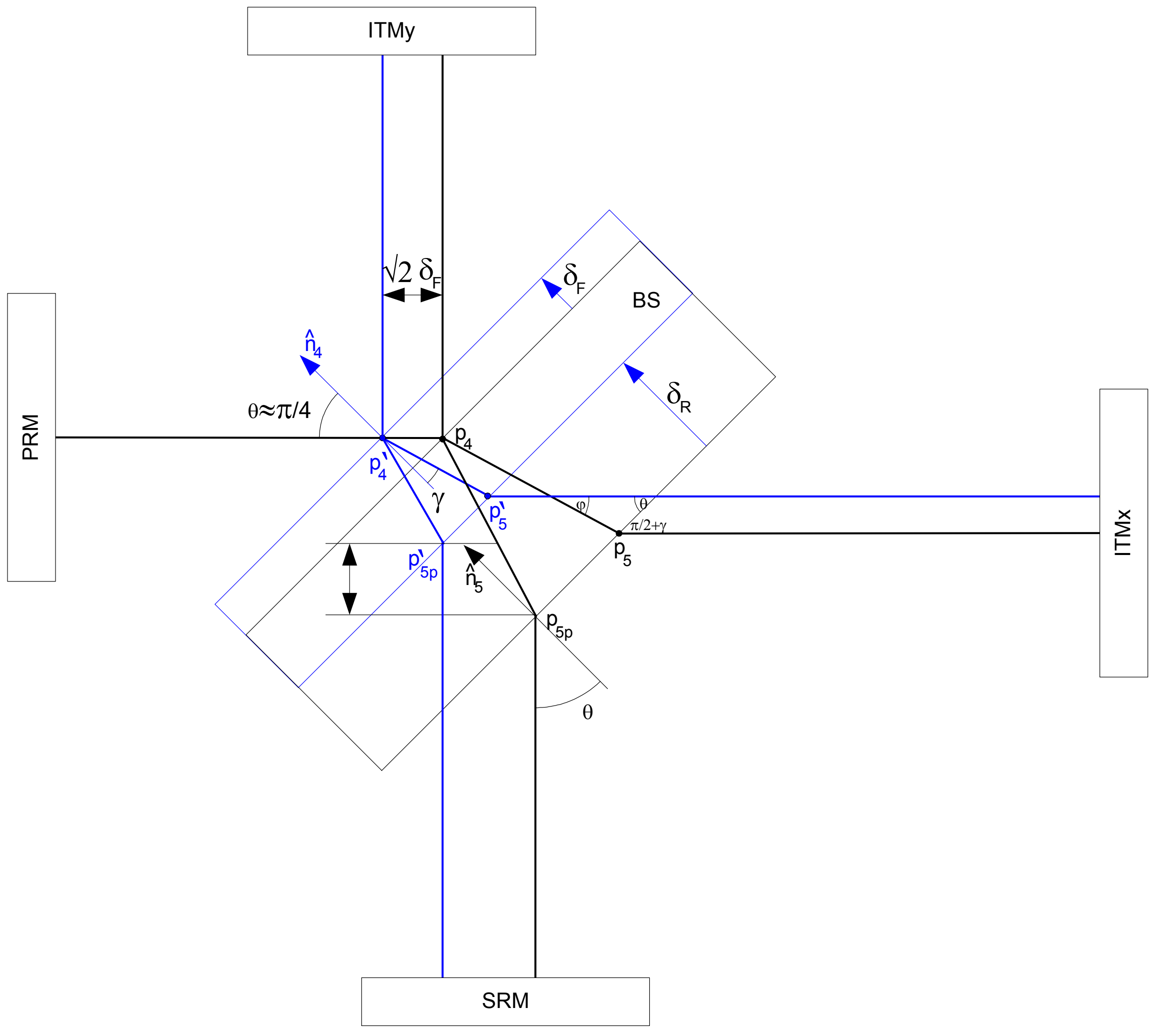

Figure 8. Power recycling cavity with vertically displaced BS

Considering the SRC paths shown in the figure above, the effect of BS vertical displacement on the SRC length can be derived to be the following:

$$\frac{dl_s^+}{ds} = \frac{n_o[\tan(\beta+\alpha)-\tan(\beta)]}{\sqrt{1-\frac{1}{2n_o^2}}}\left\{1-\frac{1}{\sqrt{2}n_o}\left[\sqrt{1-\frac{1}{2n_o^2}}+\frac{1}{\sqrt{2}n_o}\right]\right\}-\frac{1}{2}\frac{dl_p^-}{ds}$$

$$\frac{dl_s^-}{ds} = \frac{dl_p^-}{ds}$$