LIGO Laboratory / LIGO Scientific Collaboration

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Loss due to anisotropic characteristics of sapphire

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1. Theory

Anisotropic materials like sapphire have more than two elastic constants which describe the relationship between stress and strain. Each of these constants has a real and imaginary part. The ratio of imaginary to real parts is a loss angle that causes mechanical loss and thermal noise. Thus, to fully characterize the thermal noise expected from sapphire mirrors, the full set of loss angles and moduli must be known.

Sapphire has trigonal symmetry and therefore has six loss angles. The energy in the crystal must be written in a positive definite format of six terms, each term corresponding to energy that is dissipated by one of the loss angles. The distribution of this elastic energy to each term when a Gaussian pressure is applied to the mirror face, in accordance with Levin's method, can be obtained through finite element modeling. Through this model, and values for the loss angles and other relevant mechanical properties, the Brownian thermal noise of sapphire mirrors can be predicted.

To obtain values for the six loss angles, modal Q's must be measured on sapphire samples. The distribution of elastic energy into each term of the energy for every measured mode must be modeled as well. When six or more modes have reliable Q's measured, the six loss angles can be solved for using the model. These loss angles may then be used in the thermal noise model.

We have performed the calculations necessary to analyze Willem's and Busby's data for this anisotropic loss discussed in T030087-00-R. The energy for sapphire may be written in a positive definite expression after applying a change of coordinates that evidences all the symmetric features of the crystal in the stiffness matrix. Traditionally, the stiffness matrix of a trigonal crystal is written as follows:

$$[C] = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & 0 & 0 \\ & c_{11} & c_{13} & -c_{14} & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & \textit{symmetric} & & & c_{44} & -c_{14} \\ & & & & & \frac{c_{11}-c_{12}}{2} \end{pmatrix} \quad (1)$$

After applying a coordinate transformation, we obtain the symmetrized matrix shown in Equation 2. The advantage of such a stiffness matrix becomes apparent when we try to express the energy as a positive definite quantity. Note that the strains we refer to from now on are calculated in the coordinate system we obtain after the coordinate transformation and do not correspond to the strains we would use for calculating the

energy provided the stiffness matrix C. We omit the primes of the new coordinate system from now on for the sake of brevity.

$$[C'] = \begin{pmatrix} c_{11}+c_{12} & \sqrt{2}c_{13} & 0 & 0 & 0 & 0 \\ & c_{33} & 0 & 0 & 0 & 0 \\ & & c_{11}-c_{12} & \sqrt{2}c_{14} & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & \textit{symmetric} & & & c_{44} & \sqrt{2}c_{14} \\ & & & & & c_{11}-c_{12} \end{pmatrix} \quad (2)$$

The strain energy of any crystal can be calculated from the following formula:

$$E = \frac{1}{2} c_{ij} \varepsilon_i \varepsilon_j \quad (3)$$

Where c_{ij} are the components of the stiffness matrix and ε_i and ε_j are the components of strain in matrix form. Using this formula with the stiffness coefficients from C' and the strain matrix in the new reference system we can write the energy of sapphire in a positive definite form using the general formula:

$$E = \frac{1}{2} p_{ij} \left(\varepsilon_i + \frac{\left\| c'_{ij} \right\|}{c'_{ij}} (1 - \delta_{ij}) \varepsilon_j \right)^2 \quad (4)$$

where the matrix P for sapphire is (in GPa):

$$[P] = \begin{pmatrix} 492.36 & 81.32 & 0 & 0 & 0 & 0 \\ & 334.36 & 0 & 0 & 0 & 0 \\ & & 302.46 & 16.26 & 0 & 0 \\ & & & 103.46 & 0 & 0 \\ & \textit{symmetric} & & & 103.46 & 16.26 \\ & & & & & 302.46 \end{pmatrix} \quad (5)$$

Using this new energy description we identify the non-redundant coefficients of the matrix P as the significant moduli with which we can associate loss angles.

2. Calculations

Using the strain terms obtained from Dennis Coyne's models of the sapphire optics we were able to determine the energy ratios for each of the elastic moduli p for the six pink sapphire modes measured by Phil Willems and Dan Busby (see table 1). These energy ratios would theoretically help us predict the loss angles provided the loss is solely due to the bulk sapphire loss. However, fitting the data to this model results in negative and

therefore unphysical values for the loss angles (see Table 2). This is consistent with the fit in T030087-00-R which showed good results with a model using a single loss angle for bulk sapphire and a loss angle for the poorly polished barrel. It seems the lossy barrel is masking any effect from anisotropy in this data.

				Energy ratios associated with the elastic moduli					
Predicted frequency (Hz)	Measured frequency (Hz)	Mode number	Measured loss	p_{11}	p_{22}	p_{33}	p_{44}	p_{12}	p_{34}
14402	14151	4	3.87e-9	0.78%	0.11%	94.0%	0.02%	0.04%	5.06%
16240	16546	5	5.56e-9	2.59%	1.73%	87.90%	2.56%	0.12%	5.09%
16949	16732	6	4.17e-8	2.92%	1.09%	55.33%	32.72%	0.31%	7.63%
17526	17907	9	2.22e-8	24.13%	2.51%	68.15%	0.04%	1.50%	3.67%
23158	23310	13	1.72e-8	4.9%	2.65%	64.62%	21.2%	0.15%	6.47%
24154	23791	15	1.35e-8	2.59%	1.25%	29.95%	56.35%	0.21%	9.65%

Table 1: Energy distributions for an m-axis sapphire cylinder.

This data does allow for setting of limits on the individual loss angles. If each loss angle but one is assumed to be zero, and the modal Q is assumed to be due to that loss angle, a limit value can be obtained on that phi. Using the lowest phi found from this method for all the modes, a maximum phi can be quoted for each of the six loss angles. These are shown in Table 2

	Loss associated with p_{11}	Loss associated with p_{22}	Loss associated with p_{33}	Loss associated with p_{44}	Loss associated with p_{12}	Loss associated with p_{34}
Fit Values	-9.18e-7	9.96e-8	5.62e-6	1.49e-6	1.57e-6	-1.04e-4
Limit	≤9.2e-8	≤3.2e-7	≤4.12e-9	≤2.40e-8	≤4.63e-6	≤7.65e-8

Table 2: Best fit for the loss angles given the measurements available and upper boundaries established given the constraint loss angle>0.

3. Thermal noise

Brownian thermal noise in Advanced LIGO due to these limiting phis can be calculated. An FEA model of an advanced LIGO sapphire optic was constructed with a Gaussian pressure of width 6.0 cm applied to one face. This is following the method of Levin to calculate thermal noise. The elastic energy distributed into the six different energy terms is calculated for the whole optic. The ratio of energy in each energy term is then multiplied by the corresponding phi and summed across all energy terms to obtain the effective phi for thermal noise calculations.

The energy ratios from this model are shown in Table 3. Using 5 X 10^{-11} as the phi for every term except one, where the limiting value from Table 2 is used, allows the effect of these limits to be seen. The results are shown in Table 4. It is important to keep in mind these are limits, and worse case scenarios.

Percentage of energy due to each of the moduli					
p_{11}	p_{22}	p_{33}	p_{44}	p_{12}	p_{34}
14.85%	36.99%	4.01%	30.16%	9.47%	4.52%

Table 3: Energy ratios in a static analysis with pressure from a Gaussian beam.

	p_{11}	p_{22}	p_{33}	p_{44}	p_{12}	p_{34}	Goal
Eff Q (10^6)	56	8.2	201	93	2.3	121	200
Thermal noise at 100 Hz (10^{-21} m/rtHz)	5.2	14	2.9	4.0	28	3.7	3.0
BNS range	179	134	190	185	92	187	200
Relative rate	0.72	0.30	0.86	0.79	0.097	0.82	1.0

Table 4: Effect of phi limits on thermal noise, single advanced LIGO interferometer sensitivity to binary neutron star inspirals, and the relative rate of gravitational wave events.

Using the results from the fit in T030087-00-R, a predicted thermal noise in advanced LIGO from the lossy barrel can also be calculated. The same FEA model was used, and the energy ratio stored in the barrel over the bulk was found to be 3.0 X 10^{-3} within a relatively thick surface depth set by the FEA mesh. The fit in T030087-00-R got an effective barrel polish phi of 4.5 10^{-7} within roughly the same surface depth. Using these two values and a 200 million Q for the bulk sapphire, an effective Q for thermal noise calculations is found to be 157 million. This, in turn, predicts a Brownian thermal noise at 100 Hz of 3.3 X 10^{-21} m/rtHz and a binary neutron star inspiral range of 188 Mpc with very low coating thermal noise. This is in contrast to the goal of about 3.0 X 10^{-21} m/rtHz and 200 Mpc. A sensitivity change from 200 to 188 Mpc means a 20% drop in the gravitational wave event rate.

4. Conclusions

The limit that can be set on the six sapphire phis, and the corresponding thermal noise, is a worst case scenario. The most likely interpretation is that the modal Q's seen by Willems and Busby are dominated by the barrel polish, and the actual anisotropic loss in sapphire is much better than these limits. To be confident that the sapphire loss is much less, further tests are necessary.

Getting these samples repolished to a less lossy level is an obvious possibility. The samples could then be remeasured for Q, and the same analysis could be performed. Since the loss in the barrel could give rise to some excess thermal noise in advanced

LIGO, achieving good barrel polish may be necessary on its own. Both mechanical and flame polishing might be explored.

The difficulty in polishing these samples comes from their size. Working with smaller samples that can be polished to a higher standard could also provide useful data. A 150 mm diameter X 80 mm thick sapphire sample is available at MIT for Q measuring. This sample will be hung using a wire sling similar to Willem and Busby's within a week. If Q's for multiple modes can be measured, more useful limits may be set on all six loss angles in sapphire.