Field Calculations for a Power Recycled Michelson Interferometer with Fabry-Perot Arms
A First Principles Approach

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1 Introduction

In order to more fully understand the responce of the power recycled interferometer I have derived the equations for the electric field inside the interferometer. With these equations and some simplifying assumptions one can investigate the responce of the interferometer to various stimuli.

When dealing with reflection and transmision from real mirrors there can be some confusion on how to deal with relative phases acquired during the light/mirror interaction. In some of the LIGO related documents (e.g. [1], [2], [3]) the convention used is to give reflection from one side of the mirror as a negative coefficient and reflection from the other as a possitive coefficient. However, most optics documentation (e.g. [4], [5]) use a convention where the reflection is the same from either side of the optic as is the transmision, but the transmision is given an additional factor if i. Siegman [5] has a nice discussion about the two conventions in Chapter 11, page 405-6 of his book.

> For all future analyses of laser cavities and interferometers in this book, however, we will arbitratily choose the complex symmetric form $\mathbf{S} = [r, it, it, r]$, with r and t purely real, as the scattering matrix form to describe all mirrors and beamsplitters. This arbitrary choice will make no difference in any of the physical conclusions we reach about laser devices. It seems easier, however, to remember that transmission coefficients always have a factor of i associated with them than to remember which side of each mirror in a laser system is the +r and which is the -r side. [Siegman [5] page 406]

I will also use the Siegman convention as its use of symmetry does tend to make the math easier to follow. The physics is the same independent of convention. In fact the LIGO Length Sensing and Control design [6] is convention independent. I also use the convention that a propagation phase is given by $e^{-i\phi}$ and not $e^{i\phi}$, this is arbitrary but consistency is necessary.

2 Simple Fabry-Perot

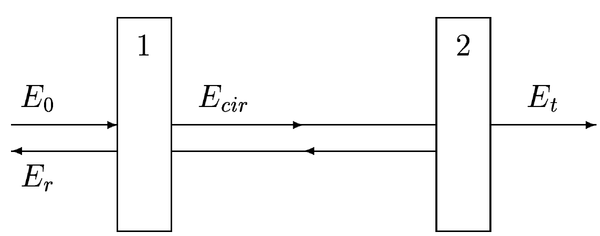

Figure 1: Simple Fabry-Perot Cavity

In the case of a generic two mirror Fabry-Perot Cavity as shown in Fig. 1 the circulating field can be represented as an infinite summation.

$$E_{cir} = it_1 E_0 + r_1 r_2 e^{-i\phi} it_1 E_0 + r_1^2 r_2^2 e^{-2i\phi} it_1 E_0 + r_1^3 r_2^3 e^{-3i\phi} it_1 E_0 + \cdots \tag{1}$$

This equals:

$$E_{cir} = \frac{it_1 E_0}{1 - r_1 r_2 e^{-i\phi}} \tag{2}$$

where ϕ is the phase collected in one round trip through the cavity (nominally $\frac{\omega}{c}2L$ with L = length of the cavity).

NOTE: The circulating field is given in terms of the E field traveling in the direction of the incident beam at mirror 1. There is a counter propagating field given by $r_2 e^{-i\phi} E_{cir}$ at mirror 1. This field is not used in these calculations.

So the cavity field equations are:

$$\begin{aligned} E_r &= r_1 E_0 + it_1 r_2 e^{-i\phi} E_{cir} \\ E_{cir} &= it_1 E_0 + r_1 r_2 e^{-i\phi} E_{cir} \\ E_t &= it_2 e^{-i\frac{\phi}{2}} E_{cir} \end{aligned}$$

which combine to give the field equations for a Fabry-Perot cavity in terms of the incident field E_0:

$$\frac{E_r}{E_0} = \frac{r_1 - r_2(r_1^2 + t_1^2)e^{-i\phi}}{1 - r_1 r_2 e^{-i\phi}} \tag{3}$$

$$\frac{E_{cir}}{E_0} = \frac{it_1}{1 - r_1 r_2 e^{-i\phi}} \tag{4}$$

$$\frac{E_t}{E_0} = \frac{-t_1 t_2 e^{-i\frac{\phi}{2}}}{1 - r_1 r_2 e^{-i\phi}} \tag{5}$$

and the power built up in the cavity can be obtained by taking the absolute magnitute squared of E_{cir}. Thus the "power build up factor" is:

$$\left|\frac{E_{cir}}{E_0}\right|^2 = \frac{t_1^2}{1 - 2r_1 r_2 \cos(\phi) + r_1^2 r_2^2} \tag{6}$$

A cavity is said to be "resonant" when the phase of the light obtained while traveling through the cavity interferes constructively with the phase of the light incident on the cavity. That is to say, a cavity is resonant when ϕ is a value providing equation 6 with the lowest denominator and thus the highest build up factor. Thus a Fabry-Perot caivty is resonant when $\cos(\phi) = 1$ and:

$$\left|\frac{E_{cir}}{E_0}\right|^2 = \frac{t_1^2}{(1 - r_1 r_2)^2} \tag{7}$$

A cavity is said to be "anti-resonant" when the phase is such that the denominator is largest, giving the smallest build up factor.

$$\left|\frac{E_{cir}}{E_0}\right|^2 = \frac{t_1^2}{(1 + r_1 r_2)^2} \tag{8}$$

3 Interferometer with Fabry-Perot Arms

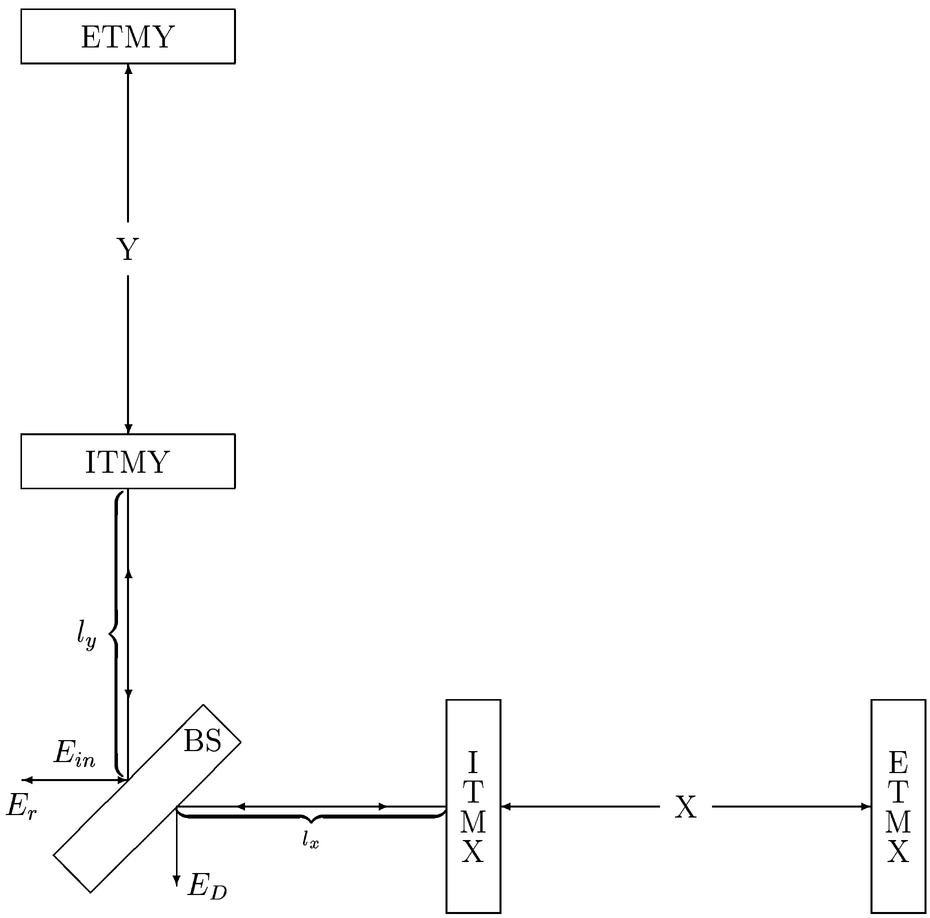

Figure 2: Simple Michelson Interferometer with Fabry-Perot Arms

Figure 2 can be simplified to Fig. 3 by treating the arm cavities as mirrors with complex reflectance given by Eq.(3) and transmission given by Eq.(5). Thus the Fabry-Perot arms X and Y are replaced with mirrors X and Y with complex reflectance r_x and r_y.

Here the input light is specified as E_{in} with the phase defined as 0 at the BeamSplitter (BS). The light will be split at the beamsplitter and reflect off the mirrors X and Y, return and interfere at the beamsplitter. If we define the distance between the beamsplitter and the Y mirror as l_y and the distance from the beamsplitter and the X mirror as l_x then the phases picked up during one round trip are $e^{-i\frac{\omega}{c}2l_x}$ or $e^{-i\frac{\omega}{c}2l_y}$ respectively. Therefore the fields for the light returning to the BS from the X,Y mirrors are:

$$\begin{array}{ll} \text{from Y} & r_y r_{bs} e^{-i\frac{\omega}{c}2l_y} E_{in} \\ \text{from X} & r_x i t_{bs} e^{-i\frac{\omega}{c}2l_x} E_{in} \end{array}$$

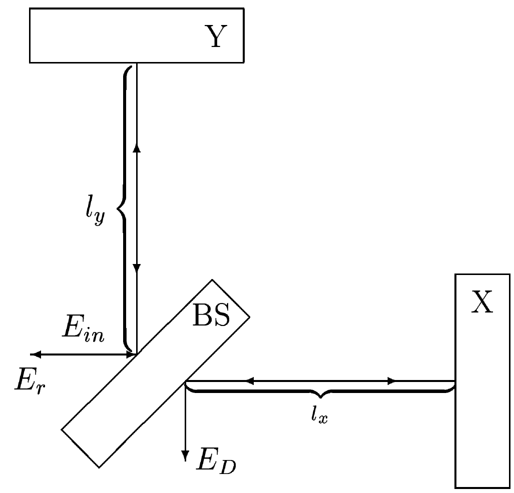

Figure 3: Simple Michelson Interferometer

The fields will interfere, sending some light back towards the input beam in the symmetric direction (the "reflected beam" E_r) and some in the anti-symmetric direction (the "dark port" E_D).

$$\frac{E_D}{E_{in}} = it_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} + ir_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x} \tag{9}$$

$$\frac{E_r}{E_{in}} = r_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} - t_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x} \tag{10}$$

Note: Special attention has and will be paid to keeping the reflection and transmission coefficients in the proper order so that the general equations which result may be used with matrix coefficients (usefull when mixing fields of different frequencies).

4 Power Recycled Interferometer

Now, for the Power Recycled Interferometer, a mirror is placed at the symmetric port to reflect most of the light being sent towards the laser, Eq.(10), back into the interferometer. This can be thought of as a simple Fabry-Perot cavity with a complex back mirror formed from the interferometer with reflection coefficient given by Eq.(10) and transmission coefficient given by Eq.(9). Fields of interest are the built up field incident upon the beamsplitter (E_{PRM}) and the field exiting the "dark" port (E_D). The field inside the cavity (E_{PRM}) is given by using Eq.(4) with r_2 specified by Eq.(10) and r_1 given by the recycling mirror with the "cavity" length specified as the distance between the recycling mirror and the beamsplitter (l_{in})[1]. The "dark" port field (E_D) is given by combining Eqs.(5, 9) in a similar fashion.

[1]The distance between the Recycling Mirror (RM) and the BeamSplitter (BS) is actually the same for both E_{PRM} and the counter propagating beam.

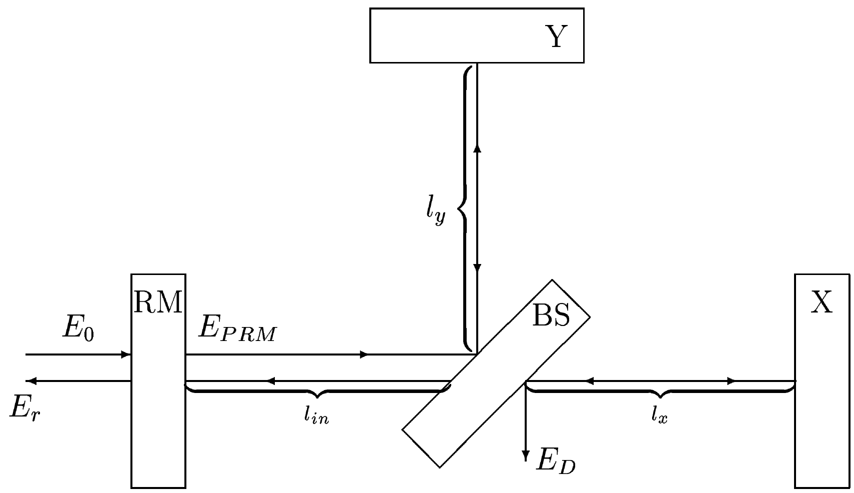

Figure 4: Power Recycled Michelson Interferometer

Thus:

$$\frac{E_{PRM}}{E_0} = \frac{it_{rm}}{1 - r_{rm}(r_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} - t_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x})e^{-i\frac{\omega}{c}2l_{in}}}$$

$$\frac{E_D}{E_0} = \frac{-t_{rm}(it_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} + ir_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x})e^{-i\frac{\omega}{c}l_{in}}}{1 - r_{rm}(r_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} - t_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x})e^{-i\frac{\omega}{c}2l_{in}}}$$

Now we do a simple variable redefinition. Since the l_{in} distance is common to both arms of the interferometer we add it to l_x and l_y and redefine the l_x, l_y distances.

$$l_x + l_{in} \rightarrow l_x$$

$$l_y + l_{in} \rightarrow l_y$$

The formulas now become:

$$\frac{E_{PRM}}{E_0} = \frac{it_{rm}}{1 - r_{rm}(r_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} - t_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x})} \tag{11}$$

$$\frac{E_D}{E_0} = \frac{(-it_{rm})(t_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} + r_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x})e^{i\frac{\omega}{c}l_{in}}}{1 - r_{rm}(r_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} - t_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x})} \tag{12}$$

and for completeness I also give the field returning to the laser:

$$\frac{E_r}{E_0} = \frac{r_{rm} - (r_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} - t_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x})(r_{rm}^2 + t_{rm}^2)}{1 - r_{rm}(r_{bs}r_y r_{bs}e^{-i\frac{\omega}{c}2l_y} - t_{bs}r_x t_{bs}e^{-i\frac{\omega}{c}2l_x})} \tag{13}$$

At this point one can define a Power Recycled Michelson Interferometer with Fabry-Perot Arms by using Eq.(3) for r_x and r_y. I will not do this substitution in general form as it provides ample opportunity for error, makes the equations even less understandable and provides no further insight into the operation of the interferometer.

5 A Perfect Interferometer

5.1 Common and Differential cavity lengths of the PRM

In order to obtain a feel for how the interferometer is locked some simplifying assumptions are needed. This will allow us to check that the calculations performed in sections 2 through 4 are correct as well as help us to understand how certain feedback signals are obtained and how to best feed them back to the interferometer. In sections 2 through 4 I took special care to maintain the proper order of operations for reflection and transmission through the optics. In this section I will assume that the mode matching from cavity to cavity is perfect and therefore will not consider scattering into higher order modes. Thus the reflection and transmission coefficients are scalars and therefore commute. Also for simplification I will consider the beamsplitter to be a perfect optic (i.e. $r_{bs} = t_{bs} = \frac{1}{\sqrt{2}}$) and that the arms are identical (i.e. $r_x = r_y = r_a$). Thus Eqs.(11, 12) become:

$$
\begin{aligned}
\frac{E_{PRM}}{E_0} &= \frac{it_{rm}}{1 - r_{rm}(\frac{1}{2}r_a e^{-i\frac{\omega}{c}2l_y} - \frac{1}{2}r_a e^{-i\frac{\omega}{c}2l_x})} \\
\frac{E_D}{E_0} &= \frac{-it_{rm}(\frac{1}{2}r_a e^{-i\frac{\omega}{c}2l_y} + \frac{1}{2}r_a e^{-i\frac{\omega}{c}2l_x})e^{i\frac{\omega}{c}l_{in}}}{1 - r_{rm}(\frac{1}{2}r_a e^{-i\frac{\omega}{c}2l_y} - \frac{1}{2}r_a e^{-i\frac{\omega}{c}2l_x})}
\end{aligned}
$$

which simplify to

$$
\frac{E_{PRM}}{E_0} = \frac{it_{rm}}{1 - \frac{1}{2}r_{rm}r_a(e^{-i\frac{\omega}{c}2l_y} - e^{-i\frac{\omega}{c}2l_x})} \tag{14}
$$

$$
\frac{E_D}{E_0} = \frac{\frac{-i}{2}t_{rm}r_a(e^{-i\frac{\omega}{c}2l_y} + e^{-i\frac{\omega}{c}2l_x})e^{i\frac{\omega}{c}l_{in}}}{1 - \frac{1}{2}r_{rm}r_a(e^{-i\frac{\omega}{c}2l_y} - e^{-i\frac{\omega}{c}2l_x})} \tag{15}
$$

where

$$
r_a = \frac{r_{itm} - r_{etm}(r_{itm}^2 + t_{itm}^2)e^{-i\frac{\omega}{c}2L}}{1 - r_{itm}r_{etm}e^{-i\frac{\omega}{c}2L}} \tag{16}
$$

At this point the almost sinusoidal form of these equations becomes obvious, the complication being that $l_x \neq l_y$. Defining the new variables $l_p = l_x + l_y$ and $l_m = l_x - l_y$ and performing the algebra gives:

$$
\begin{aligned}
e^{-i\frac{\omega}{c}2l_x} + e^{-i\frac{\omega}{c}2l_y} &= e^{-i\frac{\omega}{c}l_p}(e^{-i\frac{\omega}{c}l_m} + e^{-i\frac{\omega}{c}l_m}) \\
&= e^{-i\frac{\omega}{c}l_p}(2\cos(\frac{\omega}{c}l_m)) \qquad (17) \\
e^{-i\frac{\omega}{c}2l_x} - e^{-i\frac{\omega}{c}2l_y} &= e^{-i\frac{\omega}{c}l_p}(e^{-i\frac{\omega}{c}l_m} - e^{-i\frac{\omega}{c}l_m}) \\
&= e^{-i\frac{\omega}{c}l_p}(-2i\sin(\frac{\omega}{c}l_m)) \qquad (18)
\end{aligned}
$$

This allows for the description of the fields in terms of the Common (l_p) and Differential[2] (l_m) cavity lengths. Combining Eq.(14) with Eq.(18) as well as combining Eq.(15) with

[2] A small l is used to denote PRM lengths as a large L is reserved for the arm cavity lengths, which is not dealt with at this time.

Eqs.(17, 18) gives:

$$\frac{E_{PRM}}{E_0} = \frac{it_{rm}}{1 + ir_{rm}r_a e^{-i\frac{\omega}{c}l_p}\sin(\frac{\omega}{c}l_m)} \tag{19}$$

$$\frac{E_D}{E_0} = \frac{-it_{rm}r_a e^{-i\frac{\omega}{c}(l_p - l_{in})}\cos(\frac{\omega}{c}l_m)}{1 + ir_{rm}r_a e^{-i\frac{\omega}{c}l_p}\sin(\frac{\omega}{c}l_m)} \tag{20}$$

Which can be multiplied by their corresponding complex conjugate to give the power at that location as a fraction of the incident power on the cavity.

$$\left|\frac{E_{PRM}}{E_0}\right|^2 = \frac{t_{rm}^2}{1 + r_{rm}^2 r_a^2 \sin^2(\frac{\omega}{c}l_m) + 2r_{rm}r_a\sin(\frac{\omega}{c}l_m)\sin(\frac{\omega}{c}l_p)} \tag{21}$$

$$\left|\frac{E_D}{E_0}\right|^2 = \frac{t_{rm}^2 r_a^2 \cos^2(\frac{\omega}{c}l_m)}{1 + r_{rm}^2 r_a^2 \sin^2(\frac{\omega}{c}l_m) + 2r_{rm}r_a\sin(\frac{\omega}{c}l_m)\sin(\frac{\omega}{c}l_p)} \tag{22}$$

5.2 Resonance Conditions

In this section I look at the resonance conditions for the whole interferometer for both the carrier and the first order phase modulated sidebands used to mix with the carrier to give the feedback to keep the arms locked. Since the Pound-Drever-Hall locking technique requires that the carrier be resonant in the cavity while the sidebands are not, this is the desired resonance condition for the arm cavities. Referencing section 2 these conditions are specified in Eqs.(4 - 8). Thus for resonance we let $e^{-i\phi} \to 1$ or $\cos(\phi) \to 1$. However we want the sidebands to be almost anti-resonant[3] which means that $e^{-i\phi} \to -1$. This means that r_a from Eq.(3) can be approximated as -1 for a resonant carrier and 1 for the sidebands when numerical values from Table 1 are used. Giving us the arm resonance approximations:

$$\begin{aligned} \text{Carrier}: &\quad r_a \approx -1 \\ \text{Sideband}: &\quad r_a \approx 1 \end{aligned} \tag{23}$$

Now of course in order for the sidebands to be incident upon the arm cavities at all, they must be resonant in the power recycled Michelson. However, in order to provide maximum signal for the feedback to the arms we want the maximum amount of sideband light to leak out of the dark port. Thus taking a look at section 3 along with the assumptions for this section we can see that:

$$\frac{E_D}{E_{in}} = \frac{i}{2}r_a(e^{-i\frac{\omega}{c}l_y} + e^{-i\frac{\omega}{c}l_x})$$

so as $r_a \to 1$ for the sidebands

$$\left|\frac{E_D}{E_{in}}\right|^2 \to \cos^2(\frac{\omega}{c}l_m)$$

[3]If the first order sidebands are exactly anti-resonant then all the even order sidebands are resonant.

Now for maximum transmssion of sideband light we want $\left|\frac{E_D}{E_{in}}\right| \to t_{rm}$. This gives a restriction on the static differential PRM length l_m, which is commonly refered to as the PRM "asymmetry". Giving us our dark port resonance condition of

$$\begin{aligned} \text{Carrier :} &\quad \sin(\tfrac{\omega}{c} l_m) \approx 1 \quad \cos(\frac{\omega}{c} l_m) \approx 0 \\ \text{Sideband :} &\quad \sin(\tfrac{\omega}{c} l_m) \approx r_{rm} \quad \cos(\frac{\omega}{c} l_m) \approx t_{rm} \end{aligned} \tag{24}$$

This leaves the resonance condition for the power recycled Michelson itself which are obvious by combining Eqs.(19 or 21) with Eqns.(23 and 24).

$$\begin{aligned} \text{Carrier : } \sin(\frac{\omega}{c} l_p) \approx 1 &\quad \text{Sideband : } \sin(\tfrac{\omega}{c} l_p) \approx -1 \quad \text{With Arms Resonant} \\ \text{Carrier : } \sin(\frac{\omega}{c} l_p) \approx -1 &\quad \text{Sideband : } \sin(\tfrac{\omega}{c} l_p) \approx -1 \quad \text{With Arms } \textbf{NOT} \text{ Resonant} \end{aligned} \tag{25}$$

Note the 180° phase change in Power Recycled Michelson resonance condition when the arms are unlocked vs. when they are locked. As locking the Full IFO requires first locking the PRM without the arms (so that light is then incident upon the arms) and THEN locking the arms this requires inverting the l_p feedback signal the moment the arms resonate. This creates a need for some good electronics and some excelent programming on the part of those working on getting the instrument functioning.

5.3 Buildup Factors

When I take the nominal parameters for the LIGO Washington 2k interferometer specified in Table 1 and use them in Eqs.(6, 21 and 22) along with the resonance conditions specified in section 5.2 I get the buildup factors shown in Table 2.

Parameter	Value
Laser Power in TE00 Mode	6 (W)
Resonant SB frequency	29.50588 (MHz)
Resonant SB modulation depth	0.45
Distance RM to BS	3.022 (m)
Distance RM to inline ITM	9.528 (m)
Distance RM to offline ITM	9.828 (m)
Note: The dark port resonance conditions were used instead of these distances and frequencies due to the extreme length sensitivity involved	
BS Reflectance	0.5
BS Transmittance	0.5
RM Reflectance	0.969
RM Transmittance	0.028
RM Loss	1e-3
ITM Reflectance	0.971825
ITM Transmittance	0.0281
ITM Loss	75e-6
ETM Reflectance	0.999925
ETM Tranmittance	5e-6
ETM Loss	70e-6
Note: These are Power coefficients for the amplitude coefficients take the square root	

Table 1: Nominal Parameters for WA 2k Interferometer

Field Frequency		Field	Build-up
Arms:	carrier	$\left\|\frac{E_{cir}}{E_{in}}\right\|^2$	138.9
	sideband	$\left\|\frac{E_{cir}}{E_{in}}\right\|^2$	0.007
Remembering that the buildup factor for the arm cavities is with respect to the field incident upon the arm cavity $\|E_{in}\| = \frac{1}{\sqrt{2}} \|E_{PRM}\|$			
PRM:	carrier	$\left\|\frac{E_{PRM}}{E_0}\right\|^2$	114.7
	sideband	$\left\|\frac{E_{PRM}}{E_0}\right\|^2$	29.1
Dark Port:	carrier	$\left\|\frac{E_D}{E_0}\right\|^2$	0.000
	sideband	$\left\|\frac{E_D}{E_0}\right\|^2$	0.816

Table 2: Build-up Factors for Carrier and Resonant Sidebands in a Perfect Recycled Interferometer

References

[1] Peter Kurt Fritschel. *Techniques for Laser Interferometer Gravitational Wave Detectors.* PhD thesis, Massachusetts Institute of Technology, 1992.

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