CALIFORNIA INSTITUTE OF TECHNOLOGY
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Laser Interferometer Gravitational Wave Observatory (LIGO) Project

To/Mail Code:	
From/Mail Code:	Daniel Sigg
Phone/FAX:	
Refer to:	LIGO-T030120-00
Date:	June 15, 2003

Subject: **Angular Instabilities in High Power Fabry-Perot Cavities**

Summary:

We show that intrinsic angular instabilities exist in suspended Fabry-Perot cavities under high operating power. If the cavity mirrors are misaligned, the beam axis of the cavity mode moves. This in general leads to a beam offset at the mirror locations which in turn generates a radiation pressure induced torque. If the beam offset is in the direction of the mirror tilting away from the cavity and if the radiation pressure induced torque is larger then the restoring torque of the torsion pendulum, the system is unstable. This is not a subtle effect for LIGO cavities. For Advanced LIGO it is better to choose negative *g*-parameters for the cavity geometry.

Radiation Pressure and Torsion Pendulum:

We can write (also see T030039) the torque introduce by a laser beam of power, P, that hits a mirror at a distance, x, away from the center as:

$$T = \frac{2Px}{c} \tag{1}$$

The differential equation of an undamped torsion pendulum is

$$\frac{d^2}{dt}\alpha = -\omega^2\alpha \tag{2}$$

and the restoring torque of a torsion pendulum can be written as:

$$T = \alpha\Theta\bar{\omega}^2 \tag{3}$$

where α is the angle, Θ is the angular moment along the vertical axis and $\bar{\omega}$ is the angular frequency in yaw.

Fabry-Perot Geometry:

The beam offsets at mirror 1 and 2 of a Fabry-Perot cavity, Δx_1 and Δx_2, can be written as function of the misalignment angles, α_1 and α_2 (Siegman, Lasers, section 19.4, pg. 768):

$$\Delta x_1 = \frac{g_2}{1-g_1g_2}L\alpha_1 + \frac{1}{1-g_1g_2}L\alpha_2$$
$$\Delta x_2 = \frac{1}{1-g_1g_2}L\alpha_1 + \frac{g_1}{1-g_1g_2}L\alpha_2 \tag{4}$$

with $g_i = 1 - L/R_i$ the g-parameters and R_i the radii of curvature of the Fabry-Perot mirrors and L the cavity length.

Cavity Dynamics and Stability:

Using Eqns. (1), (2) and (3) we can write a modified torsion pendulum equation as

$$\frac{d^2}{dt}\vec{\alpha} = -(\Omega^2 + \omega^2)\vec{\alpha} \tag{5}$$

with

$$\Omega^2 = -\frac{2PL}{\Theta c}K \quad \text{and} \quad K = \begin{pmatrix} \frac{g_2}{1-g_1g_2} & \frac{1}{1-g_1g_2} \\ \frac{1}{1-g_1g_2} & \frac{g_1}{1-g_1g_2} \end{pmatrix} \tag{6}$$

and

$$\omega = \bar{\omega}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{7}$$

We solve the problem by calculating the eigenvalues of Ω, Ω_1 and Ω_2. For stability it is required that all eigenvalues of the right hand side of Eqn. (5) are negative. We can therefore write

$$\Omega_i^2 > -\bar{\omega}^2, \tag{8}$$

or with the eigenvalues of K, k_1 and k_2,

$$k_i < \frac{\Theta c \bar{\omega}^2}{2PL}. \tag{9}$$

The eigenvalues and eigenvectors of K can be written as:

$$k_1 = \frac{g_1 + g_2 - \sqrt{4 + (g_1 - g_2)^2}}{2(1-g_1g_2)} \qquad \kappa_1 = \begin{pmatrix} \frac{-g_1 + g_2 - \sqrt{4 + (g_1 - g_2)^2}}{2(1-g_1g_2)} \\ \frac{1}{1-g_1g_2} \end{pmatrix}$$
$$k_2 = \frac{g_1 + g_2 + \sqrt{4 + (g_1 - g_2)^2}}{2(1-g_1g_2)} \qquad \kappa_2 = \begin{pmatrix} \frac{-g_1 + g_2 + \sqrt{4 + (g_1 - g_2)^2}}{2(1-g_1g_2)} \\ \frac{1}{1-g_1g_2} \end{pmatrix} \tag{10}$$

The determinant of K can be calculated as

$$\det K = -\frac{1}{1 - g_1 g_2} \tag{11}$$

and since $0 \le g_1 g_2 \le 1$ for a stable resonator, one eigenvalue is always positive and one negative. Or in other words, there exists always one stable and one intrinsically unstable solution.

Application to LIGO:

For LIGO we are using the following suspension parameters:

Table 1: LIGO suspension parameters

Parameter	Description	LIGO / Adv. LIGO	Unit
M	mirror mass (LOS)	10 / 40	kg
R	mirror radius (LOS)	0.125 / 0.155	m
h	mirror thickness (LOS)	0.1 / 0.13	m
ω	yaw angular frequency (LOS)	$2\pi\,0.5$	rad/s
Θ	angular moment (yaw/pitch)	0.0474 / 0.297	kg m^2

The arm cavities have the following parameters (Adv. LIGO parameters were chosen to make the cavity most stable):

Table 2: LIGO arm cavity parameter

Parameter	4K ifo		2K ifo		Adv. LIGO	
	Value	**Unit**	**Value**	**Unit**	**Value**	**Unit**
Length	4000	m	2000	m	4000	m
ETM curvature	7400	m	7400	m	2222	m
ITM curvature	14600	m	14600	m	2222	m
g_1	0.460		0.730		–0.800	
g_2	0.726		0.863		–0.800	
$g = g_1 g_2$	0.334		0.630		0.640	
k_1	2.40		4.86		–5.01	
k_2	–0.624		–0.556		0.556	
P_{crit}	7.3	kW	7.2	kW	198	kW

where P_{crit} is calculated from Eqn. (9) using the positive k_i.

Figure 1 shows the eigenvalues, k_1 and k_2, as function of g_1 for a constant g of 0.334 (red) and 0.630 (blue). Since the eigenvalues do not change when g_1 and g_2 are interchanged, the minima and maxima are for symmetric resonators, i.e., $g_1 = g_2$. One can see that a lower g-parameters generally yields eigenvalues of lower absolute value. Also, cavities with negative g_i are more likely to be stable. Figure 2 shows the eigenvalues as function of g for symmetric resonators.

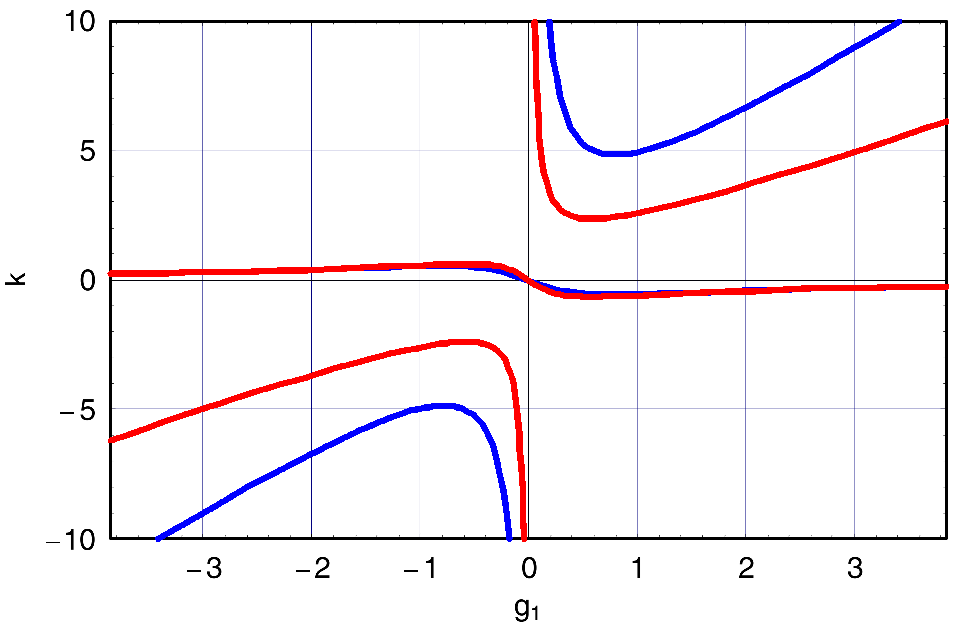

Figure 1: Eigenvalues as function of g_1 for constant g of 0.334 (red) and 0.630 (blue).

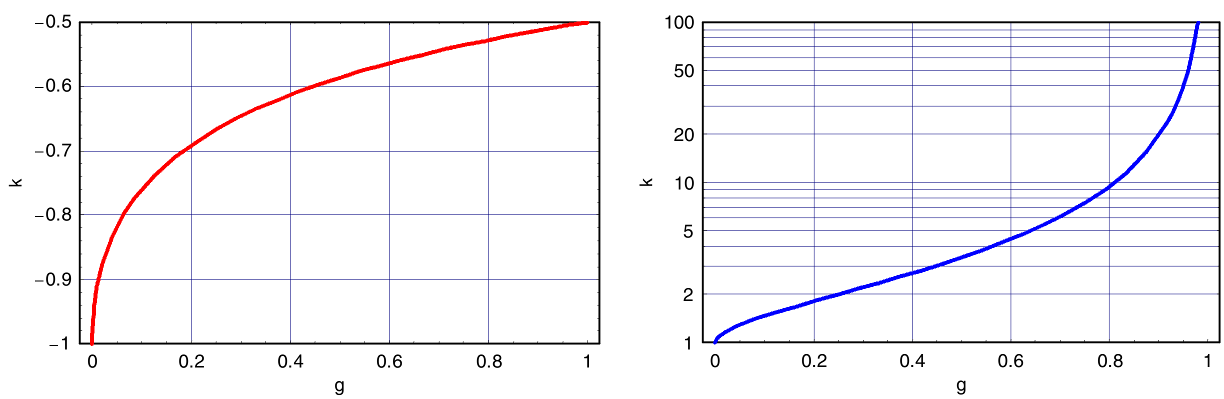

Figure 2: Eigenvalues as function of g for symmetric resonators with the stable solution to the left and the unstable one to the right.

Simple Geometrical Interpretation for Symmetric Cavities:

For a symmetric cavity it is straight forward to deduce the eigenvectors, κ_1 and κ_2. It is easy to see why one of them yields a stable configuration and other one not. Figure 3 shows a picture of the geometrical eigenmodes of a symmetric Fabry-Perot resonator. For κ_1 the cavity axis tilts around the point halfway in between the two mirrors, whereas for κ_2 the cavity axis moves up and down. In the first case the radiation pressure tends to push the misaligned mirrors back to their aligned orientation, whereas for the second case the radiation pressure pushes them further away. The second case is unstable, if the torque applied by the radiation pressure is larger than the restoring force—either by the torsion in the suspended mass or by the angular controls system.

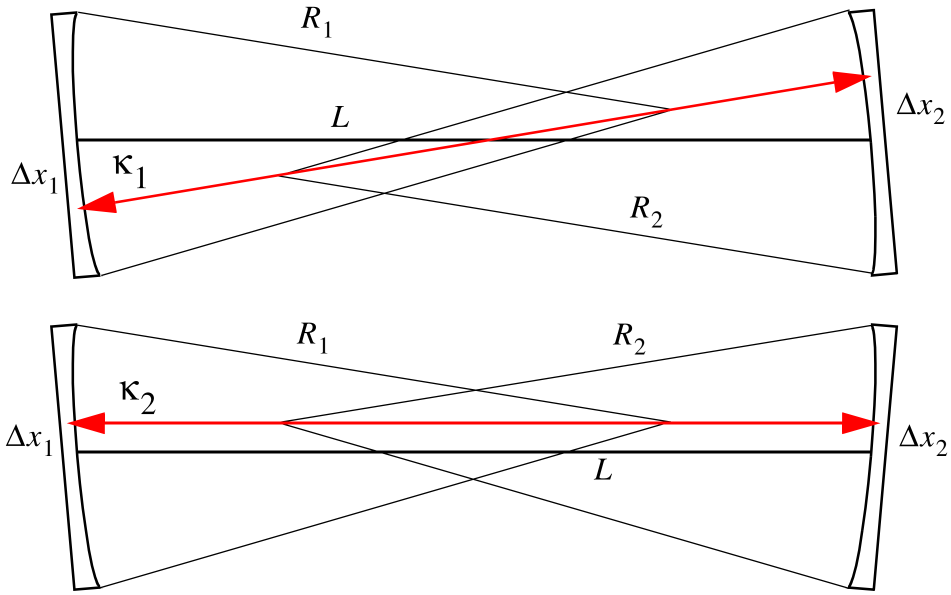

Figure 3: The geometrical eigenmodes of a symmetric resonator.

Effect of the Cavity Pole:

We assume we have diagonalized the problem and that the light storage time or cavity pole, ω_{cav}, expresses itself simply by a delayed reaction until the new cavity mode axis is established. We use Δy to denote the axis of the cavity mode and write its dependency on the geometrical axis as follows:

$$\Delta y = -\frac{1}{\omega_{\mathrm{cav}}}\frac{d}{dt}\Delta y + \Delta x \tag{12}$$

With a change of variables $\beta = \Delta y/(k_i L)$ we get the following two differential equations:

$$\begin{aligned} \beta(t) &= -\frac{1}{\omega_{\mathrm{cav}}}\frac{d}{dt}\beta(t) + \alpha(t) \\ \frac{d^2}{dt}\alpha(t) &= \Omega_i^2\beta(t) - \omega^2\alpha(t) - 2\omega\rho\frac{d}{dt}\alpha(t) \end{aligned} \tag{13}$$

where we also introduce a damping term that is controlled by the variable ρ; $\rho = 1$ corresponds to critical damping. This is fundamentally a third order linear differential equation in α. Its solutions are given by the roots of the characteristic polynomial:

$$(\Omega_i^2 - \omega^2)\omega_{\mathrm{cav}} - (\omega^2 + 2\rho\omega\omega_{\mathrm{cav}})x - (\omega_{\mathrm{cav}} + 2\rho\omega)x^2 - x^3 = 0 \tag{14}$$

Studying these roots for advance LIGO, setting $\omega_{\mathrm{cav}} = 2\pi$ rad/s and assuming critical damping shows that the stable solution, Ω_1, tends to go unstable at about the same cavity power as the unstable solution, Ω_2. Figure 4 shows the magnitude (blue) and inverse Q (red) of the solution with the largest real part for the Ω_1 case. The system is unstable if $1/Q$ becomes negative. The solution can be kept stable if the damping is increased accordingly.

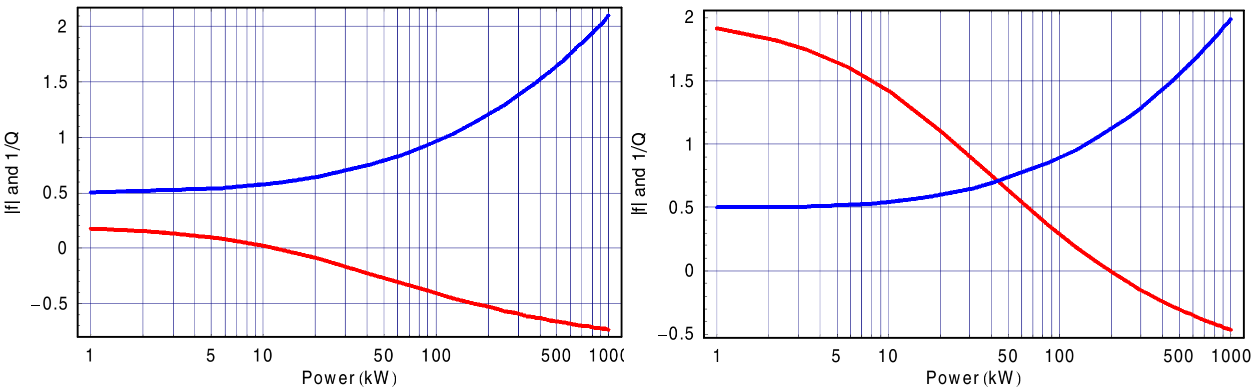

Figure 4: The magnitude (blue) and inverse Q (red) of the solution with the largest real part for the Ω_1 case. On the left the damping corresponds to a Q of 5, whereas the right side is for critical damping.