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An improved algorithm for computing gravitational wave chirps

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Templates used in a search for binary black holes and neutron stars in gravitational wave interferometer data will have to be computed on-line since the computational storage and retrieval costs for the template bank are too expensive. We propose an algorithm, based on post-Newtonian theory, that is expected to bring about an improvement in computational costs by at least a factor of 5.

I. THE POST-NEWTONIAN WAVEFORM

Post-Newtonian (PN) theory computes the gravitational wave (GW) flux $F(v)$ emitted, and the (dimensionless) relativistic binding energy $E(v)$ of, a compact binary as expansions in the gauge independent relative velocity v of the two stars. In the case of non-spinning black hole binaries the flux and energy functions are both known to order v^5 (2.5 PN) beyond the quadrupole approximation [1,2]:

$$F(v) = \frac{32\eta^2 v^{10}}{5}\left[1 - \left(\frac{1247}{336} + \frac{35\eta}{12}\right) v^2 + 4\pi v^3 - \left(\frac{44711}{9072} + \frac{9271\eta}{504} + \frac{65\eta^2}{18}\right) v^4 - \left(\frac{8191}{672} + \frac{535\eta}{24}\right)\pi v^5\right], \tag{1.1}$$

$$E(v) = -\frac{\eta v^2}{2}\left[1 - \left(\frac{9+\eta}{12}\right) v^2 - \left(\frac{81 - 57\eta + \eta^2}{24}\right) v^4\right]. \tag{1.2}$$

Gravitational waves are dominantly emitted at twice the orbital frequency of the system. The GW frequency is related to the invariant velocity v via $v = (\pi m f_{\rm GW})^{1/3}$. In the above expressions m is the total mass of the binary and $\eta = m_1 m_2/m^2$ is the symmetric mass ratio. In the quasi-circular, adiabatic approximation one uses the energy balance between the flux of gravitational waves lost from the system and the rate of decay of the binding energy of the system: $F(v(t)) = -m(dE/dt)$.

The energy balance equation can be used to set up a pair of coupled, non-linear, ordinary differential equations (ODEs) [3] to compute the orbital phase evolution $\varphi(t)$ of the binary during the adiabatic regime:

$$\frac{d\varphi}{dt} - \frac{v^3}{m} = 0, \qquad \frac{dv}{dt} + \frac{F(v)}{mE'(v)} = 0. \tag{1.3}$$

where we have made use of the relation between the orbital freqeuncy and the GW frequency, viz, $f_{\rm orb}(t) = \dot\varphi(t)/(2\pi) = f_{\rm GW}(t)/2 = v^3/(2\pi m)$ and $E'(v)$ denotes the v-derivative of $E: E'(v) = dE/dv$. In the *restricted* PN approximation, where the amplitude of the waveform is kept to the lowest PN order, while the phase is expanded to the highest PN order known, the GW radiation emitted by a binary at a distance r from Earth and sensed by an interferometric antenna, is described by the waveform

$$h(t) = \frac{4C\eta m}{r} v^2(t) \cos[\phi(t)], \tag{1.4}$$

where $\phi(t) = 2\varphi(t)$ is the gravitational wave phase. C is a constant that takes values in the range $[0, 1]$ depending on the relative orientation of the source and the antenna. It has an r.m.s (averaged over all orientations and wave polarisations) value of $2/5$.

The stationary phase approximation to the Fourier transform of the waveform in Eq. (1.4) is given by

$$\begin{aligned} \tilde{h}(f) &\equiv \int_{-\infty}^{\infty} h(t) e^{2\pi i f t} dt \\ &= \frac{4C\eta m}{r} \int_{-\infty}^{\infty} \frac{v^2(t)}{2} \left[e^{i\psi_f^+(t)} + e^{i\psi_f^-(t)} \right] dt \\ &\simeq \frac{2C\eta m}{r} \frac{v^2(t_f)}{\sqrt{\dot{f}_{\rm GW}(t_f)}} e^{i[\psi_f(t_f) - \pi/4]}, \end{aligned} \tag{1.5}$$

where $\psi_f^{\pm} = 2\pi f t \pm \phi(t)$, an overdot denotes a derivative w.r.t. t and t_f is the stationary point of the Fourier phase, $\psi_f^-(t)$ when $f \geq 0$ and $\psi_f^+(t)$ when $f \leq 0$, namely $\dot{\psi}_f(t_f) = 0$. Indeed, t_f turns out to be the instant when the GW frequency $f_{\rm GW}$ is numerically equal to the Fourier frequency $f : f = \dot{\phi}(t_f)/(2\pi) = f_{\rm GW}(t_f)$. One can solve for t_f by inverting the PN expansion of $f_{\rm GW}(t)$.

On substituting for the stationary point t_f, and consistently using the available PN expansions of the flux and energy functions one gets the usual stationary phase approximation (uSPA) to the inspiral signal

$$\psi_f(t_f) = 2\pi f T_C - \phi_C + 2\int_{v_f}^{\infty} dv (v_f^3 - v^3) \frac{dE/dv}{F} = 2\pi f T_C - \phi_C + \Psi(f; m, \eta) \tag{1.6}$$

where $v_f \equiv v(t_f) = (\pi m f)^{1/3}$. T_C is the (nominal) time of coalescence and ϕ_C is the phase at $t = T_C$, both of which can be set to zero. $\Psi(f)$ can be found by substituting, in the above equation, for the flux and energy from Eqs. (1.1) and (1.2), re-expanding the ratio $E'(v)/F(v)$ and integrating term-by-term. The resulting Fourier phase can be conveniently expressed as $\Psi(f; m, \eta) = \sum_0^4 \Psi_k(f) \tau_k$, where Ψ_k are functions only of the Fourier frequency and τ_k are the so-called (dimensionless) PN *chirp times* given in terms of the binary mass parameters

$$\begin{aligned} \tau_0 &= \frac{5}{256\eta v_0^5}, \ \tau_1 = 0, \ \tau_2 = \frac{5}{192\eta v_0^3} \left(\frac{743}{336} + \frac{11}{4}\eta \right), \\ \tau_3 &= \frac{\pi}{8\eta v_0^2}, \ \tau_4 = \frac{5}{128\eta v_0} \left(\frac{3058673}{1016064} + \frac{5429}{1008}\eta + \frac{617}{144}\eta^2 \right), \end{aligned} \tag{1.7}$$

with $v_0 = (\pi m f_0)^{1/3}$, f_0 is a fiducial frequency (e.g., the lower cutoff of the antenna response) and the Ψ_k are given, in terms of the scaled frequency $\nu \equiv f/f_0$, by:

$$\Psi_0 = \frac{6}{5\nu^{5/3}}, \ \Psi_2 = \frac{2}{\nu}, \ \Psi_3 = \frac{-3}{\nu^{2/3}}, \ \Psi_4 = \frac{6}{\nu^{1/3}}. \tag{1.8}$$

In the *restricted* PN approximation the amplitude in (1.5) depends on frequency simply through a factor $f^{-7/6}$.

Just as in the time-domain, the frequency-domain phasing is also given, in the SPA, by a pair of coupled, non-linear, ODEs:

$$\frac{d\psi}{df} - 2\pi t = 0, \qquad \frac{dt}{df} + \frac{\pi m^2}{3v^2}\frac{E'(f)}{\mathcal{F}(f)} = 0, \tag{1.9}$$

From the computational point of view, solving the ODE's above is more efficient than computing the phase algebraically using Eqs.(1.6)-Eq.(1.8).

For most binaries the uSPA is sufficiently accurate. One, therefore, generates the signal directly in the Fourier domain using Eq.(1.5) or Eq.(1.9). The latter is a quicker way of generating the waveform than solving the ODEs in Eq. (1.4) and then Fourier transforming. We shall introduce below a different representation of the signal which should further speed-up signal generation by an order of magnitude.

II. VELOCITY VARIABLE FOR FREQUENCY-DOMAIN REPRESENTATION OF THE SIGNAL

The new representation that can speed up signal generation is obtained by working in the Fourier domain with the post-Newtonian expansion parameter v. Introduce a velocity-like variable $v = (\pi m f)^{1/3}$. Indeed, v is nothing but the parameter v_f in Eq.(1.6). Use v as the Fourier variable instead of f. On substituting $f = v^3/(\pi m)$, and setting the instant of coalescence T_C and the phase at ϕ_C to zero, we find

$$\Psi(f) = \sum_{k=0}^{4} \theta_k v^{k-5},$$

where the *chirp parameters* θ_k are given by

$$\theta_0 = \frac{3}{128\eta}, \; \theta_1 = 0, \; \theta_2 = \frac{5}{96\eta}\left(\frac{743}{336} + \frac{11}{4}\eta\right),$$
$$\theta_3 = -\frac{3\pi}{8\eta}, \; \theta_4 = \frac{15}{64\eta}\left(\frac{3058673}{1016064} + \frac{5429}{1008}\eta + \frac{617}{144}\eta^2\right). \tag{2.1}$$

The Fourier transform can now be written as

$$\tilde{h}(f) = \frac{Cm^2}{r}\sqrt{\frac{5\pi\eta}{384 v^7}} \exp\left[i\sum_{k=0}^{4}\theta_k v^{k-5} - i\pi/4\right], \tag{2.2}$$

Note that in this form the phase depends only on the mass ratio η and not on the total mass m[1]. One can take advantage of this feature in lowering the computational costs in generating templates in the following manner: One computes a set of (look-up) tables of $\Psi(v;\eta)$ vs v for

[1] Although the amplitude does depend on the total mass but it is irrelevant in template generation.

different values η. The Fourier phase $\Psi(f; m, \eta)$, for a binary of mass m and mass ratio η, can simply be read off from the appropriate look-up table: $\Psi(f; m, \eta) = \Psi(v = (\pi m f)^{1/3}; \eta)$. Moreover, the cost of storing a one-parameter set of tables of $\Psi(v; \eta)$ should be several orders orders of magnitude lower than the cost of storing a two-parameter family of tables $\Psi(f; m, \eta)$.

The feature discussed above is true not just up to the PN order discussed but a very general result as can be seen from Eq.(1.6): Note that as the flux and (dimensionless) energy functions depend only on the mass ratio, the dependence of the phasing formula on the total mass comes only through the velocity variable v. If we do not substitute $v = (\pi m f)^{1/3}$ and work instead with the velocity Fourier variable, the total mass will not appear in the phasing of the Fourier domain formula.

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[1] L. Blanchet, T. Damour, B.R. Iyer, C.M. Will and A. Wiseman, Phys. Rev. Lett. **74**, 3515 (1995)

[2] L. Blanchet, Phys. Rev. **D54**, 1417 (1996).

[3] T. Damour, B.R. Iyer and B.S. Sathyaprakash, *Efficient templates to search for binary inspiral signals* (2000).