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	Report	
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Abstract

We derive the gravitational wave (GW) emission of a spinning cylinder (bar) in the Newtonian quadrupole approximation.

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We derive the gravitational wave (GW) emission of a cylinder (bar) spinning counterclockwise about the positive z direction in the quadrupole approximation.
The reduced mass quadrupole moment is defined as

$$I_{ij} = \int dV \rho \left[x_i x_j - \frac{1}{3}\delta_{ij}(x_1^2 + x_2^2 + x_3^2) \right] \ . \tag{1}$$

The bar have length L, radius R, and mean density ρ. We compute I_{ij} at time $t = 0$ and assume that at this point the bar's longest axis is in perfect alignment with the x-axis with its center being at $x = 0$. This simplifies the integral in Eq. (1) tremendously. We perform the integration in Cartesian coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$ in the following way:

$$I_{ij}(t=0) = \int_{-R}^{R} dz \int_{-L/2}^{L/2} dx \int_{-\sqrt{R^2-z^2}}^{\sqrt{R^2-z^2}} dy \ \rho \left[x_i x_j - \frac{1}{3}\delta_{ij}(x_1^2 + x_2^2 + x_3^2) \right] \ . \tag{2}$$

We now assume that the bar is spinning with angular velocity Ω. For any Ω and time t we can find $I_{ij}(\Omega, t)$ by imposing the rotation

$$I(\Omega, t) = U(\Omega, t)\, I\, U^T(\Omega, t) \ , \tag{3}$$

with the standard rotation matrix

$$U(\Omega, t)_{ij} = \begin{pmatrix} \cos\Omega t & -\sin\Omega t & 0 \\ \sin\Omega t & \cos\Omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \ . \tag{4}$$

The quadrupole transverse-traceless (TT) GW signal of the bar is proportional to the second time derivative of I_{ij},

$$h_{ij}^{TT} = \frac{2}{D}\frac{G}{c^4} P_{ijkl} \ddot{I}_{kl} \ , \tag{5}$$

where P_{ijkl} projects to the TT gauge and D is the distance to the bar. The explicit form of $\ddot{I}_{ij}$ is

$$\ddot{I}_{ij} = \frac{1}{6} M(L^2 - 3R^2)\Omega^2 \begin{pmatrix} -\cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & \cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} \ , \tag{6}$$

where we have set $M = \bar{\rho}\pi R^2 L$.

Dealing with the directional dependence of h_+ and $h_\times$

The two linearly independent GW polarizations h_+ and $h_\times$ that are observed by LIGOs depend not only on source parameters, but also on source distance, but also on the relative position of the observer frame to the source frame. Let θ be the observer co-latidude and ϕ be the observer azimuth in the source frame.
The sum $h_+ - ih_\times$ can be decomposed into modes using spin weighted spherical harmonics ${}^{-s}Y_{lm}$ of weight -2:

$$h_+ - ih_\times = \frac{1}{D} \sum_{\ell=2}^{\infty} \sum_{m=-\ell}^{\ell} H_{\ell m}(t)\, {}^{-2}Y_{\ell m}(\theta, \phi) \, . \tag{7}$$

The expansion parameters H_{lm} ($\ell = 2$ denotes the quadrupole) are complex functions of the retarded source time t.

In order to express H_{2m} in terms of $\ddot{I}_{ij}$, one first expresses $h_+(\theta,\phi)$ and $h_\times(\theta,\phi)$ in terms of $\ddot{I}_{kl}$, then convolves these with ${}^{-2}Y^*_{lm}$. The result is

$$H_{20}^{\rm quad} = \sqrt{\frac{32\pi}{15}}\frac{G}{c^4}\left(\ddot{I}_{zz} - \frac{1}{2}(\ddot{I}_{xx}+\ddot{I}_{yy})\right)\ , \tag{8}$$

$$H_{2\pm1}^{\rm quad} = \sqrt{\frac{16\pi}{5}}\frac{G}{c^4}\left(\mp\ddot{I}_{xz} + i\ddot{I}_{yz}\right)\ , \tag{9}$$

$$H_{2\pm2}^{\rm quad} = \sqrt{\frac{4\pi}{5}}\frac{G}{c^4}\left(\ddot{I}_{xx} - \ddot{I}_{yy} \mp 2i\ddot{I}_{xy}\right)\ . \tag{10}$$

For completeness, we give the definitions of the relevant ${}^{-2}Y^*_{lm}$:

$${}^{-2}Y_{22} = \sqrt{\frac{5}{64\pi}}(1+\cos\theta)^2 e^{2i\phi}\,, \tag{11}$$

$${}^{-2}Y_{21} = \sqrt{\frac{5}{16\pi}}\sin\theta(1+\cos\theta)e^{i\phi}\,, \tag{12}$$

$${}^{-2}Y_{20} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\,, \tag{13}$$

$${}^{-2}Y_{2-1} = \sqrt{\frac{5}{16\pi}}\sin\theta(1-\cos\theta)e^{-i\phi}\,, \tag{14}$$

$${}^{-2}Y_{2-2} = \sqrt{\frac{5}{64\pi}}(1-\cos\theta)^2 e^{-2i\phi}\,. \tag{15}$$

For example, assuming the detector is located along the rotation axis of the bar ($\theta = 0, \phi = 0$) we find:

$$h_+ = \frac{1}{D}\frac{G}{c^4}(\ddot{I}_{xx} - \ddot{I}_{yy}) \tag{16}$$

$$h_\times = \frac{2}{D}\frac{G}{c^4}\ddot{I}_{xy}\ . \tag{17}$$

Picking astrophysically meaningful bar parameters

Mass	Pick canonical protoneutron star (PNS) mass of $1.5M_\odot \times \epsilon$, $\epsilon = 0.1 - 1$
Length L	Only $L > R$ makes sense. Pick $L = 20\,\mathrm{km} - 60\,\mathrm{km}$
Radius R	Pick $R = 5\,\mathrm{km} - 20\,\mathrm{km}$. Ratios of L/R of $\gtrsim 1$ up to 6 make sense (axis ratios of up to 3 to 1).
Angular velocity $\Omega = 2\pi f$	Assume spin frequency f above 200 Hz, below 1000 Hz. Note that f will decrease as spin energy is lost to GWs.
Duration	May last from $\sim 1\,\mathrm{ms}$ to $\sim 1\,\mathrm{s}$.