



LASER INTERFEROMETER GRAVITATIONAL WAVE OBSERVATORY

*LIGO Laboratory / LIGO Scientific Collaboration*

LIGO-T1000154-v4

*LIGO*

April 2, 2010

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## Effects of small size anomalies in a FP cavity

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LIGO Science Collaboration

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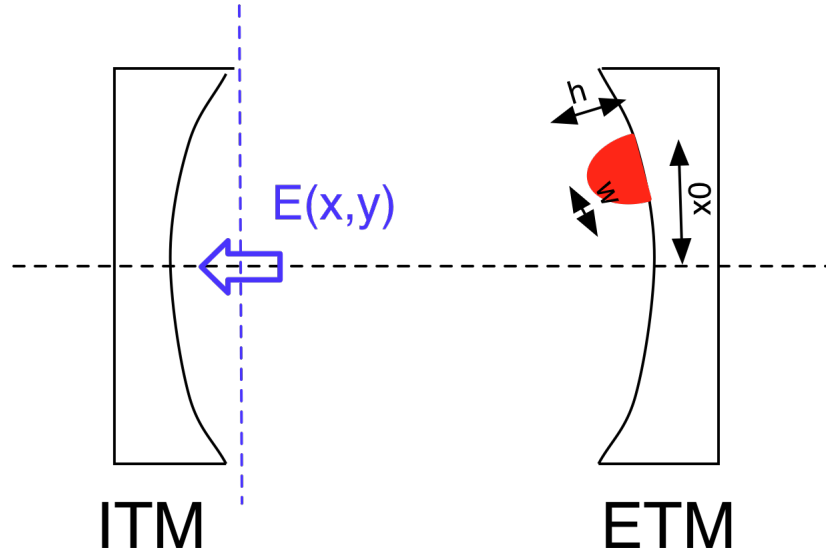
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## 1 Introduction

This note discusses the effect of small size defects on the HR surface in a FP cavity, analyzed using SIS. In this revision, (a) the loss round trip loss and (b) resonating field shapes are calculated when, in a small area on ETM, (1) the loss is not uniform or (2) the surface has a bump.

## 2 Optical system



**Figure 1. FP cavity and small anomaly**

The optical system studied is the advanced LIGO arm with an anomaly placed on ETM. The shape of the anomaly is of Gaussian shape and is placed at  $(x=x_0, y=0)$  and the anomaly is 0 out of  $2w$ , i.e.,

$$d(x,y) = h \cdot \exp\left(-\frac{(x-x_0)^2 + y^2}{w^2}\right) : \sqrt{(x-x_0)^2 + y^2} < 2w \quad (1)$$

$$= 0 : \sqrt{(x-x_0)^2 + y^2} > 2w$$

A TEM<sub>00</sub> mode field, which matches with the clean FP cavity, i.e., without the anomaly, is injected to the cavity and the stationary field is calculated.

The round trip loss without the anomaly is 0.6ppm, due to the finite apertures of ITM and ETM.

Field shapes shown below are the one on ITM surface coming from ETM, shown by an arrow in Fig.1. The effect is very small and fields called “deviation” is defined as

$$\delta(x,y) \equiv \frac{E(x,y)}{E(0,0)} - \frac{E_0(x,y)}{E_0(0,0)} \quad (2)$$

where  $E_0(x,y)$  is the stationary state field without the anomaly and  $E(x,y)$  is the one with the anomaly.  $E_0$  is a almost clean TEM<sub>00</sub> mode, and  $\delta$  is the structure added by the anomaly, which is normalized by the amplitude at the center.

### 3 Loss anomaly

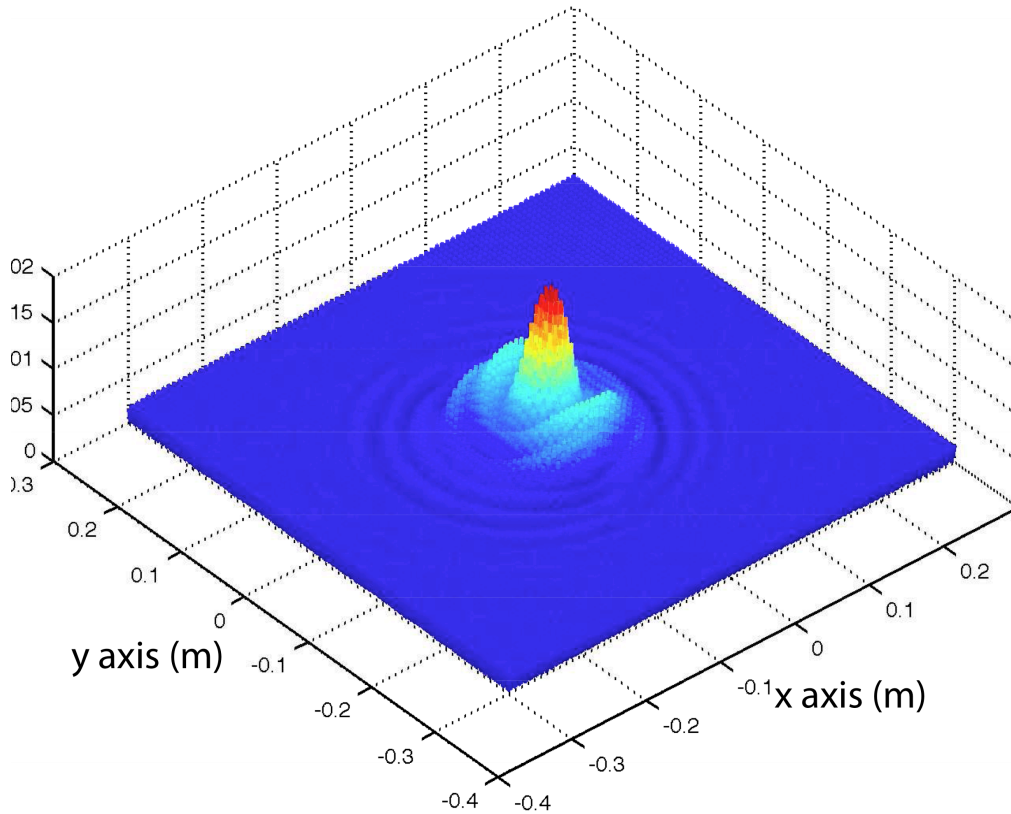
The effect of loss due to a point defect is calculated using a localized loss with the Gaussian shape. When the loss is of the Gaussian shape, the total loss on one bounce is calculated to be

$$Loss \approx P_{cav} \left( \frac{w_a}{w_{beam}} \right)^2 2h \cdot \exp\left(-\frac{2x_0^2}{w_a^2}\right) \quad (3)$$

when the size of the anomaly,  $w_a$ , is much smaller than the beam size,  $w_{beam}$ .

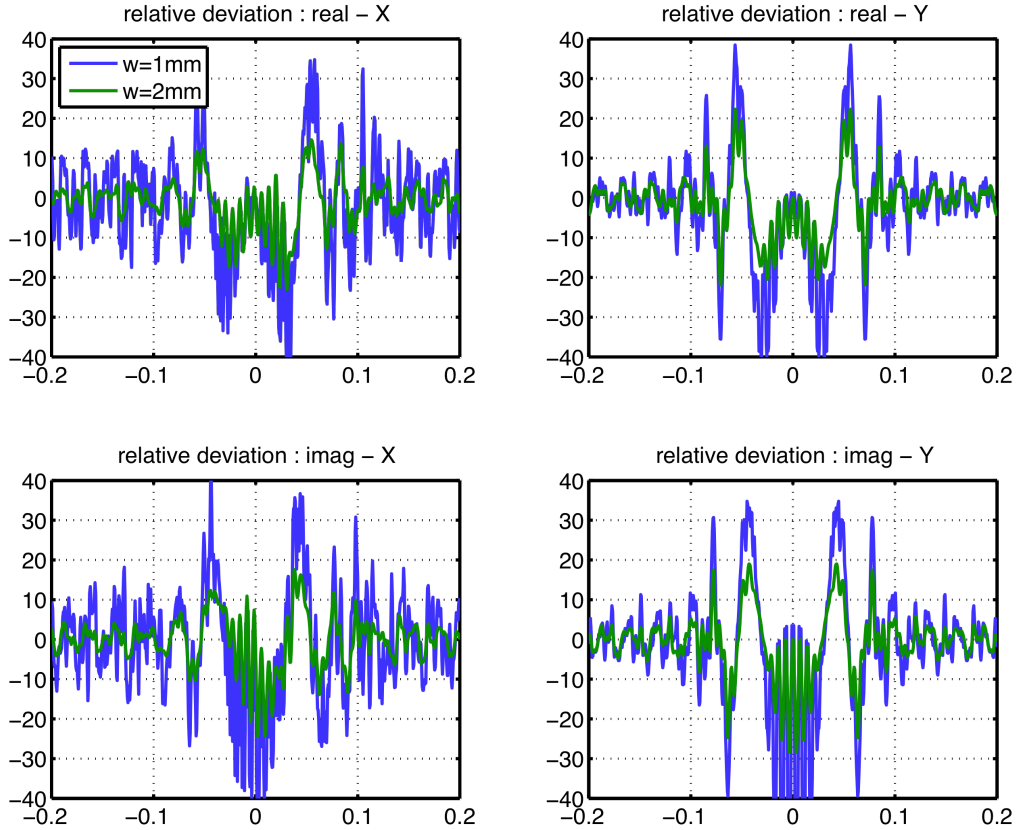
FFT-based simulation cannot simulate effects caused by point structures, which is smaller than the FFT grid size, which is a fraction of mm at the smallest.

In order to see how good the approximation is, the round trip loss was calculated using three sets of  $(w_a, h)$  which give same loss values:  $(w_a, h) = (2\text{mm}, 0.0025)$ ,  $(1\text{mm}, 0.01)$ ,  $(0.5\text{mm}, 0.04)$ . For all cases,  $x_0 = 2\text{cm}$ , and  $w_{beam} = 6.2\text{cm}$ . The losses calculated using these values were 4.8ppm, 4.7ppm and 4.8ppm, which is comparable to the analytic point loss using Eq.(3), 4.2ppm.



**Figure 2. Deviation with loss anomaly**

Fig.2 is the deviation field when a loss anomaly with  $(w_a, h) = (2\text{mm}, 0.0025)$  is placed at  $x_0=2\text{cm}$ . As is seen from the figure, the effect of a localized anomaly affects the entire beam surface. The same effect is observed when there is a localized bump, which is discussed in the following section.



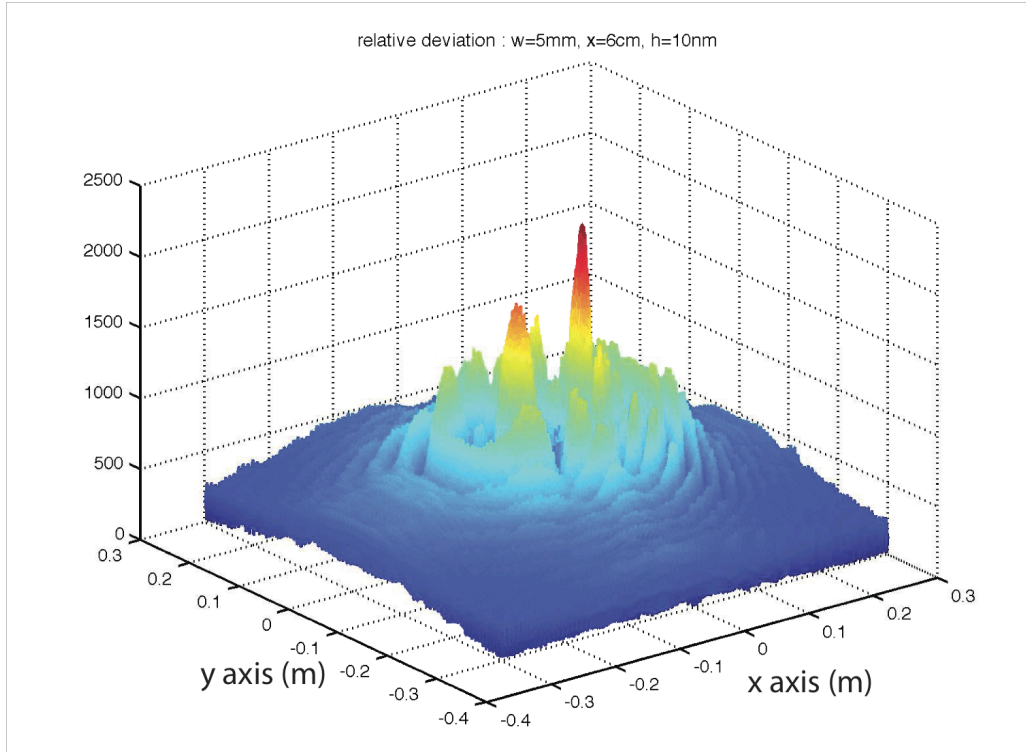
**Figure 3. Deviations of real and imaginary component (units in ppm)**

Fig.3 shows the real and imaginary components of the deviations. E.g., the top-left plot is real part of  $\delta(x,0)$ . The spatial wavelengths of the structure is determined by the location of the anomaly, i.e., 2cm in the case, both in x and y directions.

The point scattering loss measured at Caltech lab using a small size laser ( $200 \mu\text{m}$ ) was mostly less than 100ppm. On ETM, this point scattering loss will scatter out  $< (200 \mu\text{m}/6.2\text{cm})^2 \times 100\text{ppm} \sim 10^{-9}$ . This is over 1000 times smaller than the example case used to create plots in Fig.3. So the deviation by anomalous loss will change the field amplitude only by  $10^{-7}$ .

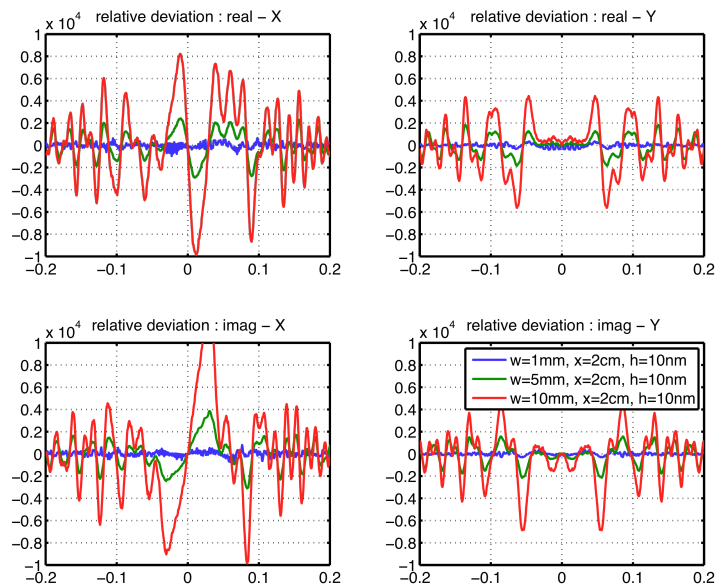
It is necessary to understand when there are many point scatterings how they affect collectively.

### 4 Shape anomaly



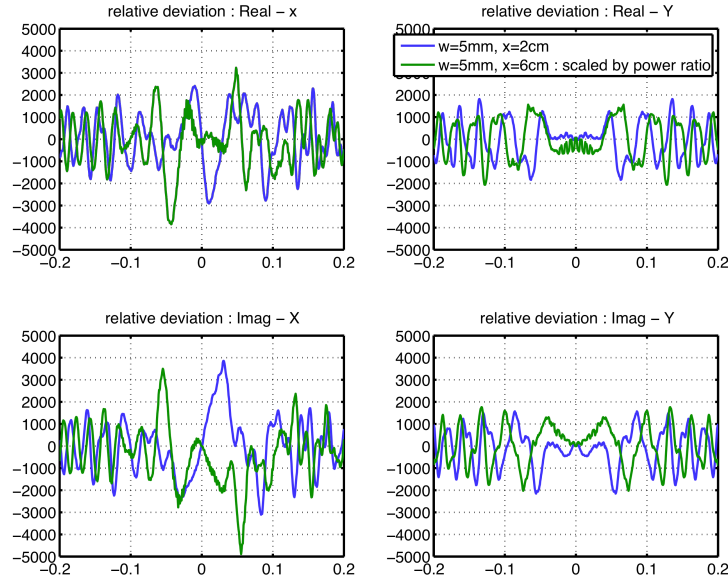
**Figure 4. Deviation with shape anomaly**

Fig.4 is the deviation when a Gaussian bump with height of 10nm, width of 5mm at location (x=6cm, y=0). As is observed above, the bump affects the entire beam, not localized around the bump.



**Figure 5. Deviations with different widths**

Fig.5 shows the real and imaginary part of deviations when the dumps are placed at 2cm away from the center. Three lines show deviations with different sizes, 1mm, 5mm and 10mm, with same height of 10nm. For each width, the round trip loss is 3.5ppm, 68ppm and 210ppm. The round trip loss is roughly proportional to the square of the height, and if the height is 1nm, the loss values are  $\sim 1/100$  of these values. These loss values and deviation magnitudes are much larger than the case of the effect of the loss anomaly.



**Figure 6. Deviations with different bump locations**

Fig. 6 shows the deviations when a bump (width 5mm, height of 10nm) is placed at 2cm and at 6cm. The blue line ( $x_0=2\text{cm}$ ) shows shorter spatial wavelength structure than the green line ( $x_0=6\text{cm}$ ).

## 5 Thermal deformation in a small region

Muzammil calculated thermal surface shape deformation by the absorption in a small region. Small beams heared at ( $x=2\text{cm}, y=0\text{cm}$ ) and the total absorption is 0.425W or 0.5ppm at the full aLIGO arm power. The beam size chosen are 0.5mm, 1mm and 2mm.

Fig. 7 and Fig.8 shows the surface deformation when the absorption is  $1/100$  of 0.5ppm. Fig.7 is the 3D plot for the beam size is 1mm, and Fig.8 compares the shapes for the beam size of 1mm and 2mm. For a larger beam size, the surface deformation is also broader.

Fig.9 shows the field distortion defined in Eq.(2) when 1mm beam is absorbed at  $0.01 \times 0.5\text{ppm}$  rate. As were discussed in previous sections, the point absorption distorted the field in the entire are, not just a limited region.

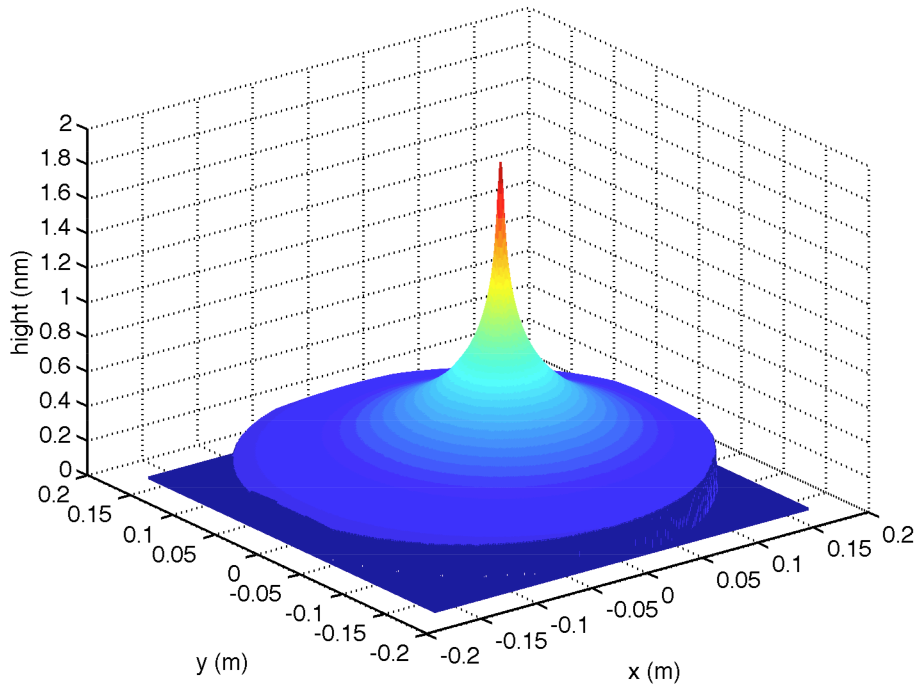


Figure 7 Surface deformation in 3D (beam size 1mm)

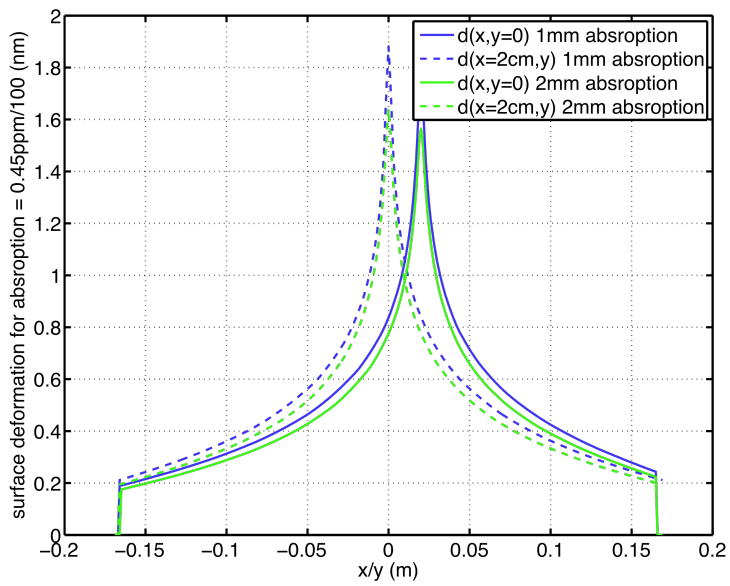
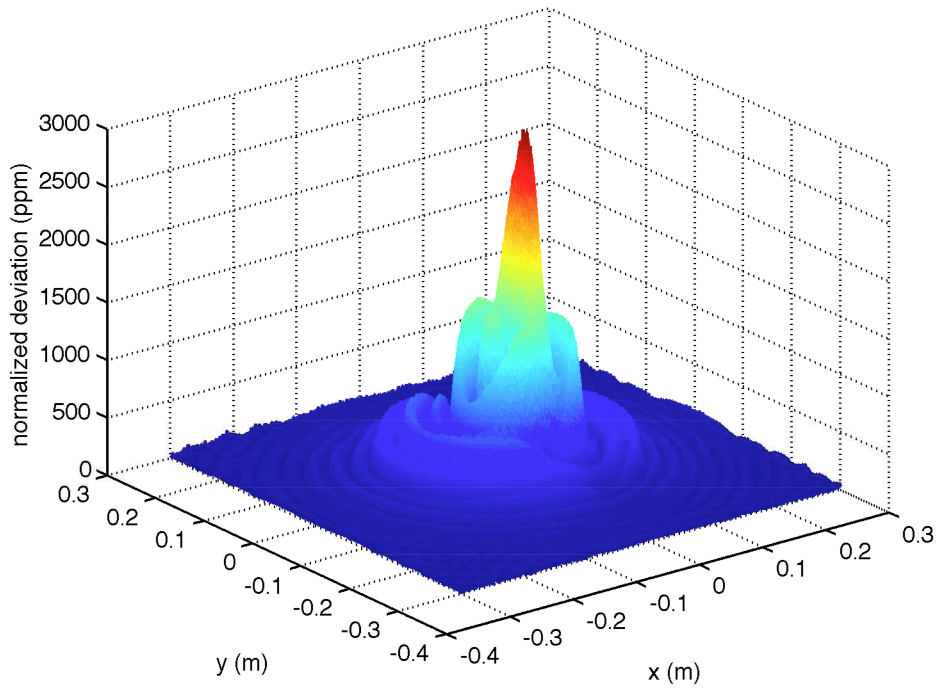
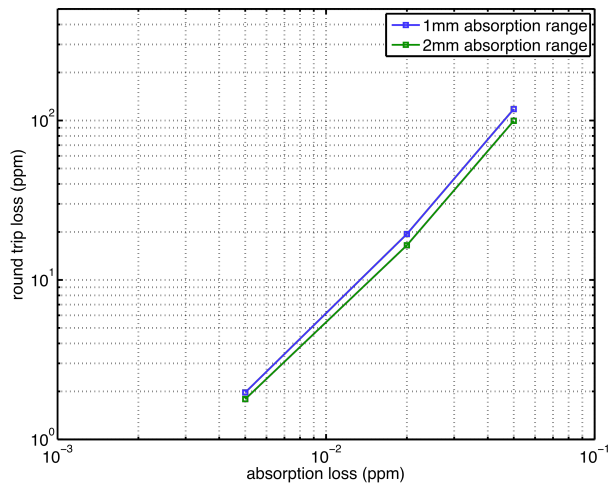


Figure 8 Surface deformation in 2D



**Figure 9 Field on ITM with 1mm heating, 1/100x0.5ppm absorption**

Fig.10 shows the relation between the absorption loss vs round trip loss. As is seen from Fig.8, the surface distortion is in a region of a several mm with a height of a few nm when the absorption is 1/100 of 0.5ppm. This induces round trip loss of a few ppm. This is consistent with the loss calculated in the previous section using a Gaussian shape.



**Figure 10 Absorption vs round trip loss**



When this calculation was done, the transmittance and reflectance of ETM was changed in the small region to reflect the local loss due to absorption. But the round trip loss is almost unaffected by the change of the optical quantities (R and T), but is determined by the surface shape. This is also consistent with the generic calculation in Sec. 3.

## 6 Point scattering using Huygen's Integral

[ The content of this section is a rewrite of Jean-Yves Vinet's note. ]

The Huyhen's integral of the field propagation from  $z_0$  to  $z$  is expressed as follows.

$$E(x, y, z) \equiv \frac{i}{\lambda} \iint dx_0 dy_0 E_0(x_0, y_0, z_0) \frac{\exp(-ik\rho)}{\rho} \cos\theta$$

$$\Delta x = x - x_0, \Delta y = y - y_0, L = z - z_0, k = 2\pi / \lambda \quad (4)$$

$$\rho = \sqrt{\Delta x^2 + \Delta y^2 + L^2}, \cos\theta = \frac{L}{\rho}$$

When a Gaussian field is reflected by a point defect, the reflected field is written as follows:

$$E_0(x_0, y_0, z_0) = TEM00(z_0) \cdot \exp(2ikf(x_0, y_0))$$

$$TEM00(z_0) = \sqrt{\frac{2}{\pi}} \frac{1}{w_0} \exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right) \quad (5)$$

For simplicity, the waist position is on the reflection surface and  $f$  is the point defect.

When inserting Eq.(5) to Eq.(4), the reflected field can be written as follows keeping only the first order of  $f$ :

$$E(x, y, z) = F_0(x, y, z) + dF(x, y, z) \quad (6)$$

$$F_0(x, y, z) = \frac{i}{\lambda} \iint dx_0 dy_0 TEM00(z_0) \frac{\exp(-ik\rho)}{\rho} \cos\theta \quad (7)$$

$$dF(x, y, z) = \sqrt{\frac{2}{\pi}} \frac{1}{w_0} \frac{i}{\lambda} \iint dx_0 dy_0 (\exp(2ikf) - 1) \exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right) \frac{\exp(-ik\rho)}{\rho} \cos\theta$$

$$\approx \sqrt{\frac{2}{\pi}} \frac{1}{w_0} \frac{i}{\lambda} \iint dx_0 dy_0 2ikf(x_0, y_0) \exp\left(-\frac{x_0^2 + y_0^2}{w_0^2}\right) \frac{\exp(-ik\rho)}{\rho} \cos\theta \quad (8)$$

$F_0$  is the unperturbed components and  $dF$  is the perturbed component by the point defect. The integral goes over the point defect where  $f$  is non-zero in a region  $O(1\mu\text{m})$ . For simplify, the defect is placed at the center of the reflection point.

With the Fresnel approximation, the Huyhen's integral becomes as follows.

$$E(x, y, z) = \exp(-ikL) \cdot E_t(x, y, z)$$

$$E_t(x, y, z) \equiv \frac{i}{L \cdot \lambda} \iint dx_0 dy_0 E_0(x_0, y_0, z_0) \exp(-ik \frac{\Delta x^2 + \Delta y^2}{2L}) \quad (9)$$

With this approximation,  $F_0$  and  $dF$  become as follows:

$$\begin{aligned}
F_0(x,y,z) &= \sqrt{\frac{2}{\pi}} \frac{1}{w_0} \frac{i}{L \cdot \lambda} \iint dx_0 dy_0 \exp(-ik \frac{\Delta x^2 + \Delta y^2}{2L}) \exp(-\frac{x_0^2 + y_0^2}{w^2}) \\
&= \sqrt{\frac{2}{\pi}} \frac{1}{w(z)} \exp(-\frac{x^2 + y^2}{w(z)^2}) \exp(i\eta(z) - i \frac{r^2}{2R(z)}) \\
&= TEM00(z)
\end{aligned} \tag{10}$$

$$\begin{aligned}
dF(x,y,z) &= \sqrt{\frac{2}{\pi}} \frac{1}{w_0} \frac{i}{L \cdot \lambda} \iint dx_0 dy_0 (\exp(2ikf) - 1) \exp(-ik \frac{\Delta x^2 + \Delta y^2}{2L}) \exp(-\frac{x_0^2 + y_0^2}{w_0^2}) \\
&\approx \sqrt{\frac{2}{\pi}} \frac{1}{w_0} \frac{i}{L \cdot \lambda} \iint dx_0 dy_0 2ikf(x_0, y_0) \exp(-ik \frac{\Delta x^2 + \Delta y^2}{2L}) \exp(-\frac{x_0^2 + y_0^2}{w_0^2})
\end{aligned} \tag{11}$$

For the typical propagation distance (~km) and the beam size (~several cm), the expression of dF can be simplified as follows, which is the Fraunhofer approximation.

$$dF(x,y,z) = \sqrt{\frac{2}{\pi}} \frac{1}{w_0} \frac{i}{L \cdot \lambda} \exp(-ik \frac{x^2 + y^2}{2L}) 2ik \iint dx_0 dy_0 \delta(x_0, y_0) \exp(ik \frac{x \cdot x_0 + y \cdot y_0}{L}) \tag{12}$$

The point loss is calculated by integrating the power of this perturbed field:

$$\begin{aligned}
point\ Loss &= \iint dx dy |dF(x,y,z)|^2 \\
&= \frac{32\pi}{w_0^2 L^2 \lambda^4} \iint dx dy \iint dx_1 dy_1 \iint dx_2 dy_2 \delta(x_1, y_1) \delta(x_2, y_2) \times \\
&\quad \exp(i2\pi \frac{x(x_1 - x_2) + y(y_1 - y_2)}{L\lambda}) \\
&= \frac{32\pi}{w_0^2 L^2 \lambda^4} \iint dx_1 dy_1 \iint dx_2 dy_2 f(x_1, y_1) f(x_2, y_2) (L\lambda)^2 \delta(x_1 - x_2) \delta(y_1 - y_2) \\
&= \frac{32\pi}{w_0^2 \lambda^2} \iint dx_0 dy_0 f(x_0, y_0)^2 \\
&= 32\pi (\frac{a}{w_0})^2 (\frac{h}{\lambda})^2
\end{aligned} \tag{13}$$

which comes out to be  $3.9 \times 10^{-5}$  ppm with  $a = 2\mu\text{m}$ ,  $w_0 = 6\text{cm}$ ,  $h = 20\text{nm}$  and  $\lambda = 1.064\mu\text{m}$ .

The loss is proportional to the power hitting the point defect, and PL is the value estimated using the power at the origin. For N defects per  $\text{mm}^2$  region, the total loss is estimated to be the following.

$$\begin{aligned}
totalLoss &= \iint dx dy PL \cdot \exp\left(-\frac{2(x^2 + y^2)}{w_0^2}\right) \times N(1/m^2) \\
&= PL \cdot \frac{\pi}{2} w_0^2 N(1/m^2) \\
&= 16\pi^2 \left(\frac{h}{\lambda}\right)^2 a^2 N(1/m^2) \\
&= \left(\frac{4\pi h}{\lambda}\right)^2 a^2 N(1/m^2) \\
&= 0.22 \times N(1/mm^2) ppm
\end{aligned} \tag{17}$$

## 7 Summary

Effects caused by localized anomalies in a FP cavity are studied using SIS. Localized anomalies can disturb the entire field. The effect of shape anomaly seems to be more problematic than the effect of the loss anomaly.

The surface deformation due to local heating was calculated using the surface deformation map by Muzzamil. If the point absorption rate is more than 1% of the nominal absorption rate, the round trip loss may become too large to affect the arm performance.

## 8 References

1. H. Yamamoto, "SIS (Stationary Interferometer Simulation) manual", LIGO-T070039