Technical Note	LIGO-T0900647-02-I	2009/12/18
	Ray optics calculations of alignment matrices	
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1 Introduction

The location of the mode of a resonant cavity can be completely defined by the waist position and angle. In the case of a two mirror cavity, its a simple matter to calculate how the waist shifts with misalignment of the mirrors. For a ring cavity with three or more mirrors, there are more mirror degrees of freedom than there are cavity mode degrees of freedom and we must define a control matrix. In this note, we show a general technique for calculating the waist motion as a function of mirror alignment using ray optics. Then we apply that technique to three test cavities of interest to LIGO: 1) a flat-curved Fabry-Perot cavity, 2) a three-mirror input mode cleaner cavity, and 3) a four mirror output mode cleaner cavity.

1.1 Zemax verification

All of the results in this document have been verified using the Zemax ray-tracing program.

2 Ray optic technique

Using ray optics for cavity misalignment is the exact same procedure as any other ray tracing operation, with two twists. First, each optic in the cavity has an associated misalignment angle. For mirrors in the ray approximation, this angle adds to the traditional ray optics matrices, as shown for flat and curved mirrors in Figure 1. Second, we use the ray tracing matrix for a cavity round trip to solve for the eigenray, $\vec{x}_0$. The eigenray has the unique property that it travels around the cavity and returns to the same location it started from. Obviously, the eigenray represents the cavity mode and waist position.
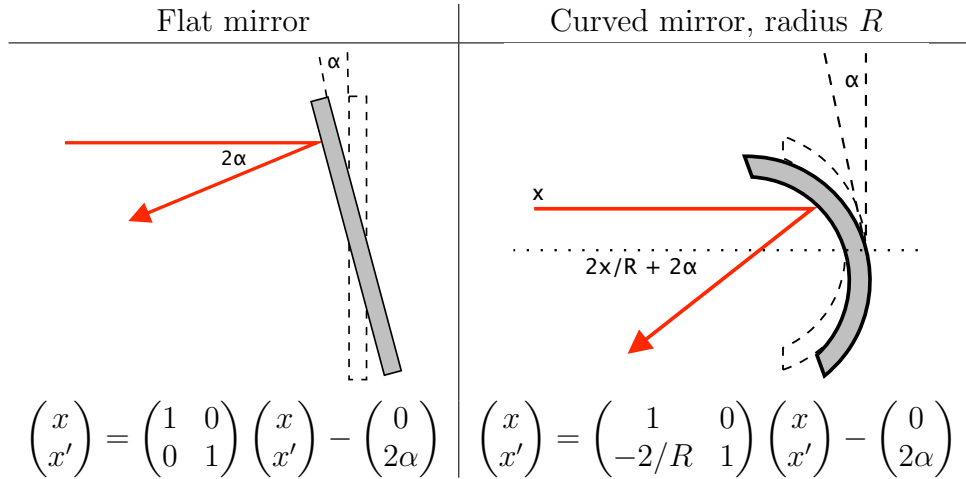

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} - \begin{pmatrix} 0 \\ 2\alpha \end{pmatrix} \quad \begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} - \begin{pmatrix} 0 \\ 2\alpha \end{pmatrix}$$

Figure 1: The effects on a ray of an optic misalignment of angle α upon reflection from flat and curved optics.

A mathematica script calculating the example matrices is included in the Appendix.
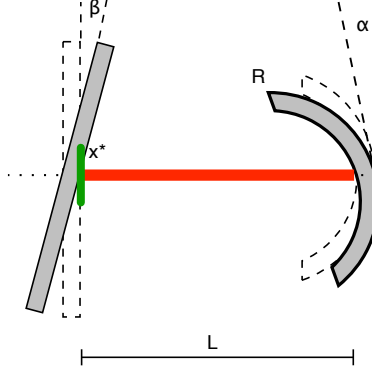
3   Two mirror Fabry-Perot

Figure 2: Layout of the flat-curved two mirror cavity. The green line labeled x^* marks the reference plane used in the calculations.

Starting from the reference plane on the left in Figure 2, we propagate a ray a distance L to the curved mirror, reflect off the curved mirror including a misalignment angle α. The ray returns a distance L to the flat mirror and picks up a second misalignment angle, β. Note that the coordinate system of each reflection, and hence the sign of the misalignment angle depends on the incoming ray. The sign is chosen such that a ray with a positive displacement is shifted towards the axis. Recalling that the eigenray $\vec{x}_0$ returns to itself after a round trip, the matrix form is:

$$\vec{x}_0 = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \vec{x}_0 + \begin{pmatrix} 0 \\ -2\alpha \end{pmatrix} \right] + \begin{pmatrix} 0 \\ -2\beta \end{pmatrix}. \tag{1}$$

Defining M as the matrix for a round trip in the cavity, the equation can be solved for the shift of the cavity waist as a function of the mirror alignment:

$$\vec{x}_0 = (1 - M)^{-1} \left[\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -2\alpha \end{pmatrix} + \begin{pmatrix} 0 \\ -2\beta \end{pmatrix} \right] \tag{2}$$

$$\vec{x}_0 = - \begin{pmatrix} R \\ 0 \end{pmatrix} \alpha - \begin{pmatrix} R - L \\ 1 \end{pmatrix} \beta \tag{3}$$

This result accords with simple geometry. Because the cavity mode must be perpendicular to the flat optic, a tilt of the curved mirror, α, has no effect on the cavity angle. The effect of α on the displacement is proportional to the radius, which is also easy to see in the paraxial ray limit. On the other hand, β, changes both the displacement and angle. The angle of the cavity mode is exactly the angle of the flat optic, while the displacement depends on the difference between the focusing of the curved mirror and the length of the cavity. When $R = L$, the cavity is in the $2fto2f$ configuration and the flat mirror alignment has no effect. For longer lengths, the cavity is unstable as evidenced by the fact that an angle misalignment shifts the cavity mode in the *opposite* direction.
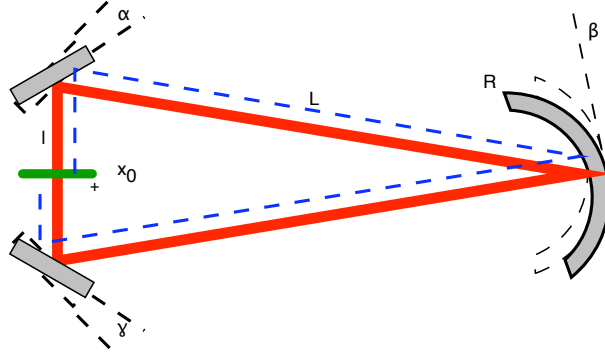
4   Three mirror input mode cleaner

Figure 3: Layout for a three mirror cavity such as the input mode cleaner. The small "+" indicates a positive displacement at the reference plane. The blue dashed line shows the effects of parity for a three mirror cavity. All of the angles are defined as positive when the mirror rotates counter-clockwise.

The three mirror cavity shown in Figure 3 is slightly more complicated in that there are three angles, α, β and γ, and two half-lengths, L and l. In general, a ray traced through a ring cavity includes *skew rays* that impact an optic off center in both x and y, thereby coupling the two dimensions on reflection. Consequently, complex geometries are easiest to solve using a ray tracing program. For a planar ring cavity such as the cavity shown in Figure 3, the horizontal and vertical planes can be separated and dealt with analytically. Three complicating factors are considered below: three dimensional rays, astigmatism and parity.

4.1   Three dimensional rays

Unlike a ray in a linear cavity, a ray in a ring cavity must explicitly include three dimensions. This is most evident in the treatment of reflections. The ray reflected lies from an optic lies in the plane formed by the incident ray and normal for the optic surface. For an optic misaligned in the plane of a ring cavity, the incident ray, optic normal, and reflected ray remain in the plane and the propagation is identical to the linear cavity. For an optic misaligned out of the plane, the reflection plane is determined by the incident ray and the vector normal to the reflecting optic, introducing a dependence on the angle of incidence. A ray incident with angle θ_{aoi} onto an optic misaligned vertically by ϕ, will generate a reflected ray with an "effective" vertical angle of $\phi_{eff} = 2\left[\cos\theta_{AOI}\right]\phi$. For the cavity shown in Figure 3, the vertical angles are modified as:

$$\alpha^V = \sqrt{\frac{L+l}{2L}}\alpha, \qquad \beta^V = \sqrt{\frac{L^2-l^2}{L^2}}\beta, \qquad \text{and} \qquad \gamma^V = \sqrt{\frac{L+l}{2L}}\gamma. \qquad (4)$$

The horizontal angles remain unchanged.

4.2 Astigmatism

In a ring cavity, the cavity ray reflects off the curved optic with a finite, sometimes large, opening angle. As with a transmissive lens, the angle of incidence introduces an astigmatism. Rays in the plane of the reflection, *planar* rays, see an effective radius of curvature of $R^p_{eff} = R\cos\theta$. Rays in the orthogonal plane see an effective radius of curvature of $R^s_{eff} = R/\cos\theta$.

4.3 Parity

The displacement and angle of a ray are inverted on reflection. However, this inversion only has a physical consequence when the cavity already has a well defined plane such as the horizontal plane, as shown schematically by the dashed blue line in Figure 3. Reflection about the vertical coordinate does not require Consequently, two solutions are required, one for horizontal shifts that includes additional minus signs:

$$\vec{x}^H_0 = -\begin{pmatrix} 1 & l+L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/R^p_{eff} & 1 \end{pmatrix}\begin{pmatrix} 1 & L+l \\ 0 & 1 \end{pmatrix}\vec{x}^H_0$$
$$+\begin{pmatrix} 1 & l+L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/R^p_{eff} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2\alpha \end{pmatrix} - \begin{pmatrix} 1 & l+L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2\beta \end{pmatrix} + \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2\gamma \end{pmatrix}, \tag{5}$$

and one for the vertical motions without the minus signs:

$$\vec{x}^V_0 = \begin{pmatrix} 1 & l+L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/R^s_{eff} & 1 \end{pmatrix}\begin{pmatrix} 1 & L+l \\ 0 & 1 \end{pmatrix}\vec{x}^V_0$$
$$+\begin{pmatrix} 1 & l+L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2/R^s_{eff} & 1 \end{pmatrix}\begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2\alpha^V \end{pmatrix} + \begin{pmatrix} 1 & l+L \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2\beta^V \end{pmatrix} + \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -2\gamma^V \end{pmatrix}. \tag{6}$$

Notice that the appropriate effective radius of curvature is used for each dimension.

4.4 Sensitivities

Solving Equations 5 and 6 for the sensitivities yields:

$$\vec{x}^H_0 = -\begin{pmatrix} L \\ \frac{R^s_{eff}-L}{R^s_{eff}-l-L} \end{pmatrix}\alpha - \begin{pmatrix} 0 \\ \frac{-R^s_{eff}}{R^s_{eff}-l-L} \end{pmatrix}\beta - \begin{pmatrix} -L \\ \frac{R^s_{eff}-L}{R^s_{eff}-l-L} \end{pmatrix}\gamma, \tag{7}$$

and

$$\vec{x}^V_0 = -\begin{pmatrix} R^s_{eff}-L \\ -\frac{L}{l+L} \end{pmatrix}\sqrt{\frac{L+l}{2L}}\alpha - \begin{pmatrix} R^s_{eff} \\ 0 \end{pmatrix}\sqrt{\frac{L^2-l^2}{L^2}}\beta - \begin{pmatrix} R^s_{eff}-L \\ \frac{L}{l+L} \end{pmatrix}\sqrt{\frac{L+l}{2L}}\gamma. \tag{8}$$

4.5 LIGO Input mode cleaner

Parameter	Symbol	Value
Total length	L	12.20 m
Short side length	l_{short}	$\approx 0.4\ m$
Radius of curvature	R	17.25 m
Curved mirror AOI	θ_{AOI}	$\approx 1°$

Table 1: LIGO input mode cleaner parameters.

As an example, we solve for the control matrix for a LIGO 4km interferometer's input mode cleaner. The parameters used here are shown in Table 1. For a ring cavity, the *perimeter*, $p = 2L$, is often used instead of the length to remove ambiguity. In the case of the input mode cleaner, the length of the long side of the triangle is $l_{long} = L - l/2 = 12\ m$.

We evaluate the sensitivity matrix from Eqs. 8, 7 numerically using the Matlab script from Appendix A:

$$\vec{x}_0^H = -\begin{pmatrix} 12 & 0 & -12 \\ 1.04 & -3.42 & 1.04 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{9}$$

and

$$\vec{x}_0^V = -\begin{pmatrix} 3.745 & 17.25 & 3.745 \\ -0.701 & 0 & 0.701 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}. \tag{10}$$

From these equations we can calculate a control matrix using Matlab's *pinv* function:

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}^H = \begin{pmatrix} -0.0417 & -0.0751 \\ 0.000 & 0.247 \\ 0.0417 & -0.0751 \end{pmatrix} \begin{pmatrix} \omega \\ \omega' \end{pmatrix}^H \tag{11}$$

and

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}^V = \begin{pmatrix} -0.0115 & 0.713 \\ -0.0530 & 0.000 \\ -0.0115 & -0.713 \end{pmatrix} \begin{pmatrix} \omega \\ \omega' \end{pmatrix}^V. \tag{12}$$

The pseudo-inverse implemented by *pinv* takes an error signal and generates a control signal, minimizing the required RMS drive. We can calculate the condition values for these matrices and find values of 4.5 and 18 for horizontal and vertical, respectively. While the normalization of these condition values is beyond the scope of this note, it is clear that the horizontal motion should be better behaved than the vertical.
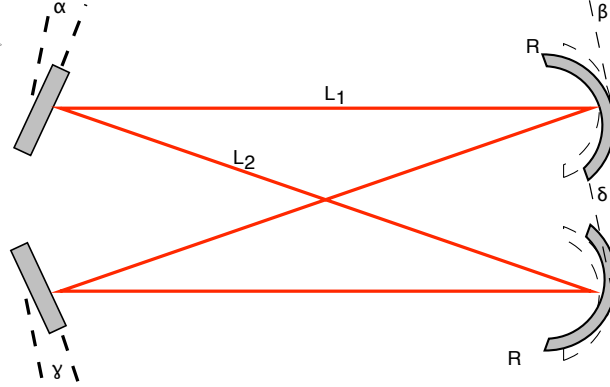
5 Four mirror output mode cleaner

Figure 4: Layout for a four mirror cavity such as the Enhanced LIGO output mode cleaner.

Our goals for analyzing the four mirror output mode cleaner are modest. We'd like to evaluate the effect of misaligned mirrors on the spot position at the curved mirrors. We'll introduce the simplifying variable $L = L_1 + L_2$ and evaluate the displacement at the plane of one of the curved mirrors. By analogy with a flat cavity, we define an effective g-factor, $g' = 1 - (L/2)/R$, that differs from a linear cavity by an extra factor of 2. Finally, because we are only interested in the order of magnitude requirements for displacements, we ignore the angle of incidence on the mirrors. The analytic solution for cavity mode motion at each of the curved mirrors is then:

$$2g_{eff}^2\, \vec{x}_0^V = \begin{pmatrix} L_1 - R \\ 1 - 2L_1/R + 2L_1/L \end{pmatrix} \alpha + \begin{pmatrix} -R \\ 1 \end{pmatrix} \beta - \begin{pmatrix} L_2 - R \\ 1 - 2L_2/L \end{pmatrix} \gamma + \begin{pmatrix} L - R \\ -1 \end{pmatrix} \delta. \quad (13)$$

Parameter	Symbol	Value
Short length	L_1	265 mm
Long length	L_2	275 mm
Radii of Curvature	R	2.0 m

Table 2: Output mode cleaner parameters used to estimate displacement at a curved mirror.

Since there are an even number of reflections, the round trip field has a parity of 1 and the horizontal and vertical solution are the same. Individual mirror perturbations will have a different sign for horizontal and vertical depending on the number of reflections between the reference plane and the optic. For the purpose of the displacement magnitude, we include only the vertical solution. The relevant parameters for the Enhanced LIGO Output Mode Cleaner are listed in Table 2. Numerically, the cavity mode displacement at one of the small optics is given by:

$$|\delta x| = \begin{pmatrix} 1.0 & 1.2 & 1.0 & 0.85 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \quad (14)$$

The OMC has a waist of $\approx 500\ \mu$m. To limit displacement to less than $1/10$ of the cavity waist requires a limit on the angular alignment of 50 μrad.

A   Input mode cleaner Matlab script

```
% function [M_h, M_v] = AlignInputModeCleaner(d_mc1mc2, d_mc1mc3, Radius)
%
% This function returns the shift of the cavity mode waist and angle as a
% function of the three input mirror angles.  Outptu is two 2x3 matrices,
% with each column representing MC1, MC2 and MC3 in turn. The first row is
% the mode displacement,  the second row is the mode angle. The first
% matrix is for Horizontal,  the second for vertical tunings.
%
% d_mc1mc2 is the distance MC1 to MC2,  and d_mc1mc3 is the distance
% MC1 to MC3.
%
% Good numbers are d_mc1mc2 = 12,  d_mc1mc3 = 0.4,  and Radius = 17.25.
%
% The math here is based on T09xxxxxx
%
%
function [M_h, M_v] = AlignInputModeCleaner(d_mc1mc2, d_mc1mc3, Radius)

try, d_mc1mc2;
catch
     d_mc1mc2 = 12;
     d_mc1mc3 = 0.4;
     Radius = 17.25;
end

%% Horizontal
 L = d_mc1mc2;
 l = d_mc1mc3/2;
 curved_AOI = asin(l / L);
 R_eff = Radius*cos(curved_AOI);

 flat_angle = (R_eff-L) / (R_eff-l-L);
 curved_angle = -R_eff/(R_eff-l-L);

 M_h = - [ L 0 -L;  flat_angle, curved_angle, flat_angle];

%% Vertical
 L = d_mc1mc2;
 l = d_mc1mc3/2;

 curved_AOI = asin(l / L);
 R_eff = Radius/cos(curved_AOI);

 flat_angle = sqrt( (L+l)/(2*L));
 curved_angle = sqrt((L^2-l^2)/L^2);


 M_v = - [ (R_eff-L)*flat_angle  R_eff*curved_angle  (R_eff-L)*flat_angle;
           -L/(l+L)*flat_angle        0              L/(l+L)*flat_angle];
```