Parameter space metric for combined diurnal and orbital motion

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This is an old document describing how the equations in the `PtoleMetric` function in LAL were derived. It is a fast approximate parameter space metric for continuous-wave searches for isolated sources. It was used in some early Einstein@Home S5 searches, and a version of this document appeared as part of David Whitbeck's 2006 Ph.D. thesis. I've finally uploaded it to the DCC, where it now has number LIGO-T0900500. —Ben

1 General formalism

We want to generate a parameter space metric to search for continuous nearly sinusoidal signals. We assume templates of the form

$$u = \exp i(2\pi f t_c + \phi) \equiv \exp i\psi. \tag{1}$$

The canonical time t_c is a function of detector time and any other parameters important in the problem, e.g. sky location (α, δ) or spin-down parameters $\{f_k\}$:

$$t_c = t_c(\alpha, \delta, f_k, \dots).$$

We will collect the template parameters together as a vector

$$\theta^\mu = (\phi, f, \alpha, \delta, f_k, \dots) \tag{2}$$

and define an inner product

$$< u|v >= \frac{1}{2T} \int_0^T dt\ u^*(t)v(t) + u(t)v^*(t). \tag{3}$$

Our detection statistic will be:

$$D(\theta, \Delta\theta) = < u(\theta)|u(\theta + \Delta\theta) >^2 . \tag{4}$$

The metric is then given by

$$g_{\mu\nu} = -\frac{1}{2} \frac{\partial^2 D}{\partial \Delta\theta^\mu \partial \Delta\theta^\nu}\bigg|_{\Delta\theta=0} \tag{5}$$

which, for our particular detection statistic and waveform, can be shown to reduce to:

$$g_{\mu\nu} = \frac{1}{T} \int_0^T \frac{\partial \psi}{\partial \Delta\theta^\mu}\bigg|_{\Delta\theta=0} \frac{\partial \psi}{\partial \Delta\theta^\nu}\bigg|_{\Delta\theta=0} dt. \tag{6}$$

The quantity ψ is given by

$$\psi = 2\pi(f + \Delta f)(t_c + \Delta t_c) + \phi + \Delta\phi,$$

where

$$t_c + \Delta t_c = t_c(\theta + \Delta\theta).$$

The partial derivatives are as follows:

$$\frac{\partial \psi}{\partial \Delta\phi}\bigg|_{\Delta\theta=0} = 1 \tag{7}$$

$$\frac{\partial \psi}{\partial \Delta f}\bigg|_{\Delta\theta=0} = 2\pi t_c \tag{8}$$

$$\frac{\partial \psi}{\partial \Delta\alpha}\bigg|_{\Delta\theta=0} = 2\pi f \frac{\partial t_c}{\partial \alpha} \tag{9}$$

$$\frac{\partial \psi}{\partial \Delta\delta}\bigg|_{\Delta\theta=0} = 2\pi f \frac{\partial t_c}{\partial \delta} \tag{10}$$

$$\frac{\partial \psi}{\partial \Delta f_k}\bigg|_{\Delta\theta=0} = 2\pi f \frac{\partial t_c}{\partial f_k} \tag{11}$$

2 Doppler modulation only

Let's include both the Earth's spin and orbital motions, and compute the metric for

$$\theta^\mu = (\phi, f, \alpha, \delta).$$

Table 1: Accuracy of computed metric components

Component	Terms missing *beyond* leading order
$g_{\phi,\phi}$	Exact
$g_{\phi,f}$	v/c missing
$g_{\phi,\alpha}$	Exact
$g_{\phi,\delta}$	Exact
$g_{f,f}$	v/c and $(v/c)^2$ missing
$g_{f,\alpha}$	v/c missing
$g_{f,\delta}$	v/c missing
$g_{\alpha,\alpha}$	Exact
$g_{\alpha,\delta}$	Exact
$g_{\delta,\delta}$	Exact

The canonical timing function for an epicyclic (Ptolemaic) detector motion is:

$$\begin{aligned} t_c = t &+ \cos\delta\cos\alpha(R_o\cos\phi_o + R_s\cos\lambda\cos\phi_s) \\ &+ \cos\delta\sin\alpha(R_o\cos\iota\sin\phi_o + R_s\cos\lambda\sin\phi_s) \\ &+ \sin\delta(R_o\sin\iota\sin\phi_o + R_s\sin\lambda). \end{aligned} \tag{12}$$

Here R_o is an astronomical unit, R_s is the Earth's (equatorial) radius, λ is the latitude of the detector, ι is the angle between the Earth's spin and orbital angular momenta, ϕ_o is the orbital phase (measured from vernal equinox), and ϕ_s is the rotation phase (measured from detector midnight).

It is then straightforward (but tedious) to evaluate the partial derivatives above and insert them into (5) to evaluate the metric. We will make use of the small quantities

$$\frac{v_s}{c} \sim \frac{R_s\omega_s}{c}$$
$$\frac{v_o}{c} \sim \frac{R_o\omega_o}{c}$$

to evaluate each metric component only to leading non-zero order. The corresponding accuracy is then summarised in table 1. The fractional errors of order v/c propagate during the projections to lower dimensional metrics, but the fractional errors in any projected component will never be worse than v/c.

2.1 Useful definitions

These simply make the metric expression more manageable.

For any quantity Q:

$$\Delta Q \equiv Q_{\rm final} - Q_{\rm initial}. \tag{13}$$

In particular

$$\Delta\phi_o \equiv \phi_{o,\rm final} - \phi_{o,\rm initial} = \omega_o T \tag{14}$$

$$\Delta\phi_s \equiv \phi_{s,\rm final} - \phi_{s,\rm initial} = \omega_s T, \tag{15}$$

where T is the integration time, and for trig expressions

$$\Delta \sin\phi \equiv \sin\phi_{\rm final} - \sin\phi_{\rm initial} \tag{16}$$

$$A_1 = R_o \frac{\Delta\sin\phi_o}{\Delta\phi_o} + R_{\rm s}\cos\lambda \frac{\Delta\sin\phi_s}{\Delta\phi_s} \tag{17}$$

$$A_2 = R_o \cos i \frac{\Delta\cos\phi_o}{\Delta\phi_o} + R_s\cos\lambda \frac{\Delta\cos\phi_s}{\Delta\phi_s} \tag{18}$$

$$A_3 = -R_o \sin i \frac{\Delta\cos\phi_o}{\Delta\phi_o} + R_s\sin\lambda \tag{19}$$

$$A_4 = R_o \left[\frac{\sin\phi_{o,\rm final}}{\Delta\phi_o} + \frac{\Delta\cos\phi_o}{(\Delta\phi_o)^2}\right] \tag{20}$$

$$A_5 = R_s \left[\frac{\sin\phi_{s,\rm final}}{\Delta\phi_s} + \frac{\Delta\cos\phi_s}{(\Delta\phi_s)^2}\right] \tag{21}$$

$$A_6 = R_o \left[-\frac{\cos\phi_{o,\rm final}}{\Delta\phi_o} + \frac{\Delta\sin\phi_o}{(\Delta\phi_o)^2}\right] \tag{22}$$

$$A_7 = R_s \left[-\frac{\cos\phi_{s,\rm final}}{\Delta\phi_s} + \frac{\Delta\sin\phi_s}{(\Delta\phi_s)^2}\right] \tag{23}$$

$$A_8 = R_o^2 \left(1 + \frac{\Delta\sin 2\phi_o}{2\Delta\phi_o}\right) \tag{24}$$

$$A_9 = R_o R_s \left[\frac{\Delta\sin(\phi_o - \phi_s)}{\Delta\phi_o - \Delta\phi_s} + \frac{\Delta\sin(\phi_o + \phi_s)}{\Delta\phi_o + \Delta\phi_s}\right] \tag{25}$$

$$A_{10} = R_s^2 \left(1 + \frac{\Delta \sin 2\phi_s}{2\Delta\phi_s}\right) \tag{26}$$

$$A_{11} = R_o^2 \frac{\Delta \cos 2\phi_o}{2\Delta\phi_o} \tag{27}$$

$$A_{12} = R_o R_s \left[-\frac{\Delta \cos(\phi_o - \phi_s)}{\Delta\phi_o - \Delta\phi_s} + \frac{\Delta \cos(\phi_o + \phi_s)}{\Delta\phi_o + \Delta\phi_s}\right] \tag{28}$$

$$A_{13} = R_o R_s \left[\frac{\Delta \cos(\phi_o - \phi_s)}{\Delta\phi_o - \Delta\phi_s} + \frac{\Delta \cos(\phi_o + \phi_s)}{\Delta\phi_o + \Delta\phi_s}\right] \tag{29}$$

$$A_{14} = R_s^2 \frac{\Delta \cos 2\phi_s}{2\Delta\phi_s} \tag{30}$$

$$A_{15} = R_o^2 \left(1 - \frac{\Delta \sin 2\phi_o}{2\Delta\phi_o}\right) \tag{31}$$

$$A_{16} = R_o R_s \left[\frac{\Delta \sin(\phi_o - \phi_s)}{\Delta\phi_o - \Delta\phi_s} - \frac{\Delta \sin(\phi_o + \phi_s)}{\Delta\phi_o + \Delta\phi_s}\right] \tag{32}$$

$$A_{17} = R_s^2 \left(1 - \frac{\Delta \sin 2\phi_s}{2\Delta\phi_s}\right) \tag{33}$$

$$A_{18} = R_0 R_s \frac{\Delta \sin \phi_o}{\Delta\phi_o} \tag{34}$$

$$A_{19} = R_s^2 \frac{\Delta \sin \phi_s}{\Delta\phi_s} \tag{35}$$

$$A_{20} = R_0 R_s \frac{\Delta \cos \phi_o}{\Delta\phi_o} \tag{36}$$

$$A_{21} = R_s^2 \frac{\Delta \cos \phi_s}{\Delta\phi_s} \tag{37}$$

$$B_1 = A_4 + A_5 \cos \lambda \tag{38}$$

$$B_2 = A_6 \cos i + A_7 \cos \lambda \tag{39}$$

$$B_3 = A_6 \sin i + \frac{R_s \sin\lambda}{2} \tag{40}$$

$$B_4 = A_8 + 2A_9 \cos\lambda + A_{10} \cos^2\lambda \tag{41}$$

$$B_5 = A_{11} \cos i + A_{12} \cos\lambda + A_{13} \cos i \cos\lambda + A_{14} \cos^2\lambda \tag{42}$$

$$B_6 = A_{15} \cos^2 i + 2A_{16} \cos i \cos\lambda + A_{17} \cos^2\lambda \tag{43}$$

$$B_7 = -A_{11} \sin i + 2A_{18} \sin\lambda - A_{13} \sin i \cos\lambda + A_{19} \sin 2\lambda \tag{44}$$

$$B_8 = A_{15} \sin i \cos i - 2A_{20} \cos i \sin\lambda + A_{16} \sin i \cos\lambda - A_{21} \sin 2\lambda \tag{45}$$

$$B_9 = A_{15} \sin^2 i - 4A_{20} \sin i \sin\lambda + 2R_s^2 \sin^2\lambda \tag{46}$$

2.2 The metric components

The metric components can then be written rather compactly as follows.

$$g_{\phi\phi} = 1 \tag{47}$$

$$g_{\phi f} = \pi T \tag{48}$$

$$g_{\phi\alpha} = -2\pi f \cos\delta [A_1 \sin\alpha + A_2 \cos\alpha] \tag{49}$$

$$g_{\phi\delta} = 2\pi f [-A_1 \sin\delta \cos\alpha + A_2 \sin\delta \sin\alpha + A_3 \cos\delta] \tag{50}$$

$$g_{ff} = \frac{(2\pi T)^2}{3} \tag{51}$$

$$g_{f\alpha} = (2\pi)^2 f \cos\delta T [-B_1 \sin\alpha + B_2 \cos\alpha] \tag{52}$$

$$g_{f\delta} = (2\pi)^2 fT[-B_1 \sin\delta\cos\alpha - B_2 \sin\delta\sin\alpha + B_3\cos\delta] \tag{53}$$

$$g_{\alpha\alpha} = 2(\pi f\cos\delta)^2[B_4\sin^2\alpha + B_5\sin 2\alpha + B_6\cos^2\alpha] \tag{54}$$

$$\begin{aligned} g_{\alpha\delta} = 2(\pi f)^2\cos\delta[&B_4\sin\alpha\cos\alpha\sin\delta - B_5\sin^2\alpha\sin\delta \\ &-B_7\sin\alpha\cos\delta + B_5\cos^2\alpha\sin\delta \\ &-B_6\sin\alpha\cos\alpha\sin\delta + B_8\cos\alpha\cos\delta] \end{aligned} \tag{55}$$

$$\begin{aligned} g_{\delta\delta} = 2(\pi f)^2[&B_4\cos^2\alpha\sin^2\delta + B_6\sin^2\alpha\sin^2\delta \\ &+B_9\cos^2\delta - B_5\sin 2\alpha\sin^2\delta \\ &-B_8\sin\alpha\sin 2\delta - B_7\cos\alpha\sin 2\delta] \end{aligned} \tag{56}$$

2.3 Refinement for short duration observations

The function `PtoleMetric` has difficulty in calculating the metric for short duration observations. It seems that this is caused by finite accuracy errors in subtracting nearly equal quantities involving the orbital phase. To give the computer a helping hand, the following replacements are useful, where a Taylor series expansion in the orbital phase change, $\Delta\phi_o$ is used. This allows the leading order terms to be cancelled by hand.

For short duration observations the quantities A_4 and A_6 are form of subtractions involving terms of order $1/\Delta\phi_o$ to give a result of order unity. These can be recast in the form

$$A_4 = R_o\left[\frac{S_1}{\Delta\phi_o}\cos\phi_{o,\mathrm{final}} + S_2\sin\phi_{o,\mathrm{final}}\right]$$

$$A_6 = R_o\left[\frac{S_1}{\Delta\phi_o}\sin\phi_{o,\mathrm{final}} - S_2\cos\phi_{o,\mathrm{final}}\right]$$

where

$$S_1 = \frac{\Delta\phi_o}{2!} - \frac{\Delta\phi_o^3}{4!} + \ldots$$

$$S_2 = \frac{\Delta\phi_o}{3!} - \frac{\Delta\phi_o^3}{5!} + \ldots$$

The trigonometric subtractions can be recast as:

$$\frac{\Delta \sin\phi_o}{\Delta\phi_o} = S_1 \sin\phi_{o,\text{final}} + \frac{\sin\Delta\phi_o}{\Delta\phi_o}\cos\phi_{o,\text{final}}$$

which appears in A_1 and A_{18}.

$$\frac{\Delta \cos\phi_o}{\Delta\phi_o} = S_1 \cos\phi_{o,\text{final}} - \frac{\sin\Delta\phi_o}{\Delta\phi_o}\sin\phi_{o,\text{final}}$$

which appears in A_2, A_3 and A_{20}. Similarly

$$\frac{\Delta \sin 2\phi_o}{2\Delta\phi_o} = S_3 \sin 2\phi_{o,\text{final}} + \frac{\sin 2\Delta\phi_o}{2\Delta\phi_o}\cos 2\phi_{o,\text{final}}$$

which appears in A_8 and A_{15}, and S_3 is the same as S_2 with the replacement $\phi_o \to 2\phi_o$:

$$S_3 = \frac{2\Delta\phi_o}{3!} - \frac{(2\Delta\phi_o)^3}{5!} + \ldots$$

Finally

$$\frac{\Delta 2\cos\phi}{2\Delta\phi} = S_3 \cos 2\phi_{o,\text{final}} - \frac{\sin 2\Delta\phi_o}{2\Delta\phi_o}\sin 2\phi_{o,\text{final}}$$

which appears in A_{11}.

3 Spin-down only

Now include spin-down only, i.e. compute the metric for

$$\theta^\mu = (\phi, f, f_k).$$

and

$$t_c = t + \sum_{k=1} \frac{f_k}{k+1}(t-t_0)^{k+1}. \tag{57}$$

The relevant partial derivatives are now:

$$\frac{\partial\phi}{\partial\phi_0} = 1 \tag{58}$$

$$\frac{\partial\phi}{\partial f} = 2\pi t_c \tag{59}$$

$$\frac{\partial\phi}{\partial f_k} = 2\pi f \frac{\partial t_c}{\partial f_k} \tag{60}$$

Table 2: Accuracy of computed metric components for spin-down only.

Component	Terms missing *beyond* leading order
$g_{\phi,\phi}$	Exact
$g_{\phi,f}$	f_k missing
g_{ϕ,f_k}	Exact
$g_{f,f}$	f_k and f_k^2 missing
g_{f,f_k}	f_k missing
g_{f_j,f_k}	Exact

We will evaluate each metric component to leading non-zero order, making use of the smallness of $\{f_k\}$. We obtain:

$$g_{\phi\phi} = 1 \tag{61}$$

$$g_{\phi f} = \pi T \tag{62}$$

$$g_{\phi f_k} = \frac{2\pi f T^{k+1}}{(k+1)(k+2)} \tag{63}$$

$$g_{ff} = \frac{(2\pi T)^2}{3} \tag{64}$$

$$g_{f f_k} = \frac{(2\pi)^2 T^{k+2} f}{(k+1)(k+3)} \tag{65}$$

$$g_{f_j f_k} = \frac{(2\pi f T)^2 T^{j+k}}{(j+1)(k+1)(j+k+3)} \tag{66}$$

The corresponding accuracies are given in table 2.

4 Doppler modulation *and* spin-down

The canonical time can now be approximated as

$$t_c = t + \Delta t_{\mathrm{Doppler}} + \Delta t_{\mathrm{spin-down}}$$

where the two Δ terms on the right hand side are the previously computed delays due to Doppler effects and spin-down. Note that this neglects a Doppler—spin-down term which would account for the fact that the intrinsic frequency of the source would change in the time it takes the gravitational waves to propagate across the Earth's orbit. It can be shown that this is always a negligible effect.

If we continue to evaluate each metric component to leading order only, we can readily convince ourselves that very little new work need be done. The components break down into four classes:

1. Those independent of Doppler modulation and spin-down (and therefore identical to those computed in both earlier calculations):

   $g_{\phi\phi}, g_{\phi f}, g_{ff}$

2. Those identical to the Doppler modulation only case:

   $g_{\phi\alpha}, g_{\phi\delta}, g_{f\alpha}, g_{f\delta}, g_{\alpha\alpha}, g_{\alpha\delta}, g_{\delta\delta}$

3. Those identical to spin-down only case:

   $g_{\phi f_k}, g_{f f_k}, g_{f_j f_k}$

4. New Doppler modulation—spin-down components:

   $g_{\alpha f_k}, g_{\delta f_k}$

The new components are as follows:

$$g_{f_k\alpha} = \frac{(2\pi f)^2}{(k+1)T}\{-\cos\delta\sin\alpha[R_o I^o_{k,c} + R_s\cos\lambda I^s_{k,c}] + \cos\delta\cos\alpha[R_o\cos\iota I^o_{k,s} + R_s\cos\lambda I^s_{k,s}]\} \tag{67}$$

$$g_{f_k\delta} = \frac{(2\pi f)^2}{(k+1)T}\{-\sin\delta\cos\alpha[R_o I^o_{k,c} + R_s\cos\lambda I^s_{k,c}] - \sin\delta\sin\alpha[R_o\cos\iota I^o_{k,s} + R_s\cos\lambda I^s_{k,s}] + \cos\delta[R_o\sin\iota I^o_{k,s} + R_s\sin\lambda\frac{T^{k+2}}{k+2}]\} \tag{68}$$

where

$$I^o_{k,c} = \int_0^T t^{k+1}\cos\phi_o\,dt$$

$$I^o_{k,s} = \int_0^T t^{k+1}\sin\phi_o\,dt$$

and similarly for the spin quantities. These are best evaluated iteratively using the following set of equations:

$$I_{k+1,c} = \frac{T^{k+1}}{\omega}\sin\phi_{\rm final} - \frac{k+1}{\omega^2}\left\{-T^k\cos\phi_{\rm final} + kI_{k-1,c}\right\} \tag{69}$$

$$I_{-1,c} = \int_0^T \cos\phi\, dt = \frac{\Delta\sin\phi}{\omega} \tag{70}$$

$$I_{0,c} = \int_0^T t\cos\phi\, dt = T\frac{\sin\phi_{\rm final}}{\omega} + \frac{\Delta\cos\phi}{\omega^2} \tag{71}$$

$$I_{k+1,s} = -\frac{T^k}{\omega}\cos\phi_{\rm final} + \frac{(k+1)}{\omega^2}\left\{T^k\sin\phi_{\rm final} - kI_{k-1,s}\right\} \tag{72}$$

$$I_{-1,s} = \int_0^T \sin\phi\, dt = -\frac{\Delta\cos\phi}{\omega} \tag{73}$$

$$I_{0,s} = \int_0^T t\sin\phi\, dt = -\frac{T}{\omega}\cos\phi_{\rm final} + \frac{\Delta\sin\phi}{\omega^2} \tag{74}$$

5 Projecting out the Phase

The phase offset of the waveform ϕ is a nuisance parameter, and is projected out so that the template spacing will only be determined by the metric on the intrinsic parameter subspace of the full parameter space. The projected metric γ is calculated by the standard formula

$$\gamma_{\mu\nu} = g_{\mu\nu} - \frac{g_{\mu\phi}g_{\nu\phi}}{g_{\phi\phi}} \tag{75}$$

Analytically that means that since

$$g_{\mu\nu} = \langle \partial_\mu\psi\partial_\nu\psi\rangle \tag{76}$$

Then we have

$$\gamma_{\mu\nu} = \langle \partial_\mu\psi\partial_\nu\psi\rangle - \langle\partial_\mu\psi\rangle\langle\partial_\nu\psi\rangle \tag{77}$$

by virtue of (7). These projections are numerically computed in PtoleMetric.c.

Table 3: Definitions of angles, lengths, etc...

Symbol	definition
u	Signal template
f	A fixed frequency
t_c	Canonical time
t	Detector time
ϕ	Phase of signal at $t_c = 0$
ψ	Signal phase $= 2\pi f t_c + \phi$
α, δ	Source RA and dec
$\{f_k\}$	Spin-down parameters (see Eq. (57)
θ^μ	Vector of signal parameters
D	Detection statistic
T	Duration of observation
R_s	Radius of Earth
R_o	Radius of Earth's orbit
ϕ_o, ϕ_s	Earth's orbital and spin phases
λ	Polar angle giving detector latitude ($\lambda = 0$ at N pole)
ι	Misalignment of Earth's spin and orbital vectors (approx 23°).
ω_o, ω_s	Angular velocities of Earth's orbit and spin motions
$\Delta\phi_o, \Delta\phi_s$	Change in orbital, spin phases over T.