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Subject:	Dynamic Vibration Absorber		
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1. Introduction

One approach to mitigating or eliminating the risk of parametric instability, due to the coupling of optical and acoustic modes in the Advanced LIGO Fabry-Perot arm cavities, is to add passive damping to the acoustic modes by the addition of one or more passive dynamic dampers. In this technical memo we define the dynamics of the coupled test mass and absorber dynamics. The test mass frequencies of concern range from ~10 kHz to ~100 kHz.

2. Dynamic Absorber Dynamics

We simplify the dynamics of the infinite dimensional test mass as the combination of a finite set of normalized, orthogonal modes over the acoustic frequency range of interest. The effect of the addition of a set of dynamic absorbers is then treated as independent perturbations of each absorber on each test mass mode. The idealized case of a two degree of freedom coupled oscillator is depicted in Figure 1. The dynamic absorber mass, M_2, is attached to the primary mass, M_1, with a damped resilient element with complex modulus $G^*_1(\omega)$ which varies with frequency, ω. The primary mass dynamics are represented by a damped resilient element with complex modulus $G^*_2(\omega)$.
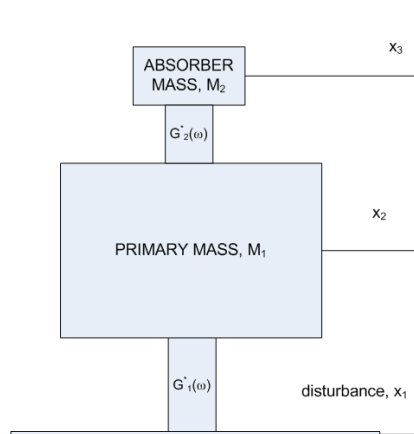

Figure 1: Simplified dynamic absorber

For the case where the primary mass has multiple modes, the absorber coupling depends upon the mode shape amplitude at the absorber location, φ, as depicted in Figure 2.
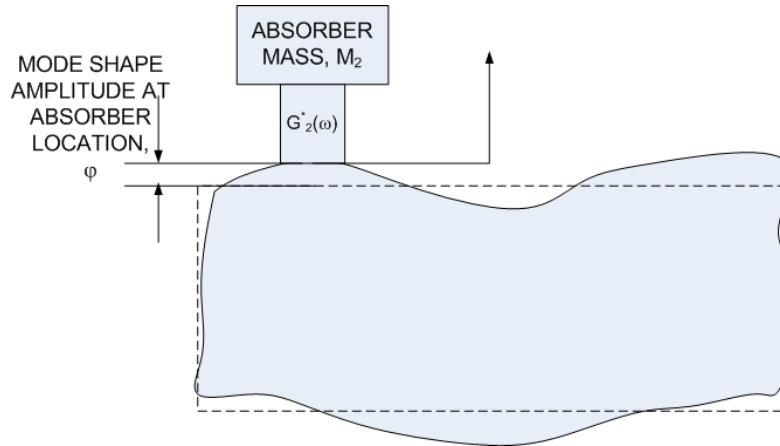

Figure 2: Effect of primary (test) mass mode shape on dynamic absorber coupling

The simplified case of a dynamic damper has been analyzed previously by many authors; Here we follow the formulation given by Snowdon[1], with the addition of the mode shape coupling factor. The equations of motion for each of the masses are as follows:

$$M_1 \frac{d^2 x_2}{dt^2} = k_1 G_1^*(\omega)(x_1 - x_2) - \varphi k_2 G_2^*(\omega)(\varphi x_2 - x_3)$$

$$M_2 \frac{d^2 x_3}{dt^2} = \varphi k_2 G_2^*(\omega)(\varphi x_2 - x_3)$$

where

φ is the mode shape amplitude at the location of the dynamic absorber

k_2 is a constant (with units of length) determined by the geometry of the resilient element of the dynamic absorber. For example if the resilient element is used directly in shear, k_2 = A/l, the ratio of the cross-sectional area to the length.

k_1 is a constant (with units of length) required for compatibility of units, but does not have a physical analog to the case where the primary mass is connected to the ground; the product $k_1 G_1^*(\omega)$ defines an effective spring constant for the mode of interest.

$$G_1^*(\omega) = G_1(\omega)\left(1 + i\delta_1(\omega)\right)$$
$$G_2^*(\omega) = G_2(\omega)\left(1 + i\delta_2(\omega)\right)$$

are the complex moduli for the resilient elements of the primary and absorber masses. The elastic modulus, G, and the loss factor, δ, should be chosen for the appropriate direction of strain (e.g. dilatation or shear).

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[1] J.C. Snowdon, Vibration and Shock in Damped Mechanical Systems, John Wiley & Sons, cr 1968, section 4.2.

In these equations of motion x_2 is the generalized coordinate (modal coordinate) for the primary mass whereas x_3 can be taken to be the physical or the generalized coordinate since the absorber is modeled as a single degree of freedom element. In order to develop the elastic and viscoelastic force due to the relative motion between the dynamic damper and the test mass, the test mass generalized motion, q_2, is transformed to a physical displacement at the dynamic absorber location, q_2, as follows:

$$q_2 = \varphi x_2$$

where φ is the amplitude of the mode shape at the absorber position. The resulting physical viscoelastic force, $k_2 G_2^*(\omega)(\varphi x_2 - x_3)$, is then transformed to a generalized force by multiplying by the mode shape coupling factor, φ.

In the above equations of motion, if we substitute $x_3' = x_3 / \varphi$, $k_2' = \varphi^2 k_2$ and $M_2' = \varphi^2 M_2$, then the equations are identical to Snowdon's (equations 4.3 and 4.4). The transfer function from disturbance to test mass motion in the presence of the dynamic absorber is then[2]:

$$T(\omega) = \left| \frac{x_2}{x_1} \right| = \sqrt{\frac{R_N + i I_N}{R_D + i I_D}}$$

where

$$I_N = \left[\Omega^2 \left(\frac{G_2(\omega_a)}{G_2(\omega)} \right) - n^2 \left(1 - \delta_1(\omega)\delta_2(\omega) \right) \right]$$

$$R_N = \left[-\Omega^2 \left(\frac{G_2(\omega_a)}{G_2(\omega)} \right) \delta_1(\omega) + n^2 \left(\delta_1(\omega) + \delta_2(\omega) \right) \right]$$

$$I_D = \left\{ -\mu\Omega^4 \left(\frac{G_1(\omega_0)}{G_1(\omega)} \right)\left(\frac{G_2(\omega_a)}{G_2(\omega)} \right) + \Omega^2 \left[n^2 \left(\frac{G_1(\omega_0)}{G_1(\omega)} \right) + \left(\frac{G_2(\omega_a)}{G_2(\omega)} \right) \right] - n^2 \left(1 - \delta_1(\omega)\delta_2(\omega) \right) \right\}$$

$$R_D = \left\{ -\Omega^2 \left[n^2 \left(\frac{G_1(\omega_0)}{G_1(\omega)} \right) \delta_2(\omega) + \left(\frac{G_2(\omega_a)}{G_2(\omega)} \right) \delta_1(\omega) \right] + n^2 \left(\delta_1(\omega) + \delta_2(\omega) \right) \right\}$$

and

$$\mu = \frac{M_1}{M_1 + \varphi^2 M_2}$$

$$n = \frac{\omega_a}{\omega_0}$$

---

[2] Snowdon has an error in his equations 4.12 through 4.15 for the real and imaginary terms of the numerator and denominator of the transfer function. Snowdon's imaginary terms are actually the real terms and the real terms are the negative of the imaginary terms.

$$\Omega = \omega / \omega_0$$

$$\omega_a^2 = \frac{k_2 G_2(\omega_a)}{M_2}$$

$$\omega_0^2 = \frac{k_1 G_1(\omega_0)}{M_1 + \varphi^2 M_2}$$

The above general equations apply for structural, viscous or viscoelastic damping. The derivation of the above equations is given in Appendix A.

In the particular case where the damping of the primary resilient element is negligible and the elastic modulus is frequency independent (as is the case with the test mass), $G_1^*(\omega) = G_1$, and the above equations reduce to the following:

$$I_N = \left[\Omega^2 \left(\frac{G_2(\omega_a)}{G_2(\omega)} \right) + n^2 \right]$$

$$R_N = n^2 \delta_2(\omega)$$

$$I_D = \left\{ -\mu \Omega^4 \left(\frac{G_2(\omega_a)}{G_2(\omega)} \right) + \Omega^2 \left[n^2 + \left(\frac{G_2(\omega_a)}{G_2(\omega)} \right) \right] + n^2 \right\}$$

$$R_D = n^2 \delta_2(\omega)(1 - \Omega^2)$$

If in addition, the complex modulus of the resilient member of the dynamic absorber is frequency independent, then $G_2^*(\omega) = G_2(1 + i\delta_2)$ and

$$I_N = \Omega^2 + n^2$$

$$R_N = n^2 \delta_2$$

$$I_D = -\mu \Omega^4 + \Omega^2 (n^2 + 1) + n^2$$

$$R_D = n^2 \delta_2 (1 - \Omega^2)$$

and the square of the transfer function is

$$T^2 = \frac{(n^2 - \Omega^2)^2 + (n^2 \delta_2)^2}{\left\{ \left[\mu \Omega^4 - \Omega^2 (1 + n^2) + n^2 \right]^2 + (n^2 \delta_2)^2 (1 - \Omega^2)^2 \right\}}$$

which is Snowdon's equation 4.27 with $\alpha = \dfrac{G_2(\omega_a)}{G_2(\omega)} \equiv 1$.

When the absorber loss factor, δ_1, is low and the absorber frequency is close to the undamped resonance of the primary mass (i.e. when n ≈ 1), then the transfer function has two peaks. As the absorber loss factor is increased, the amplitudes of the two peaks decrease to a minimum. If the absorber loss factor is increased further then the peaks merge

to a single peak of higher amplitude, i.e. there is an optimum loss factor for an absorber tuned to a single primary mass resonance (see Figure 3).
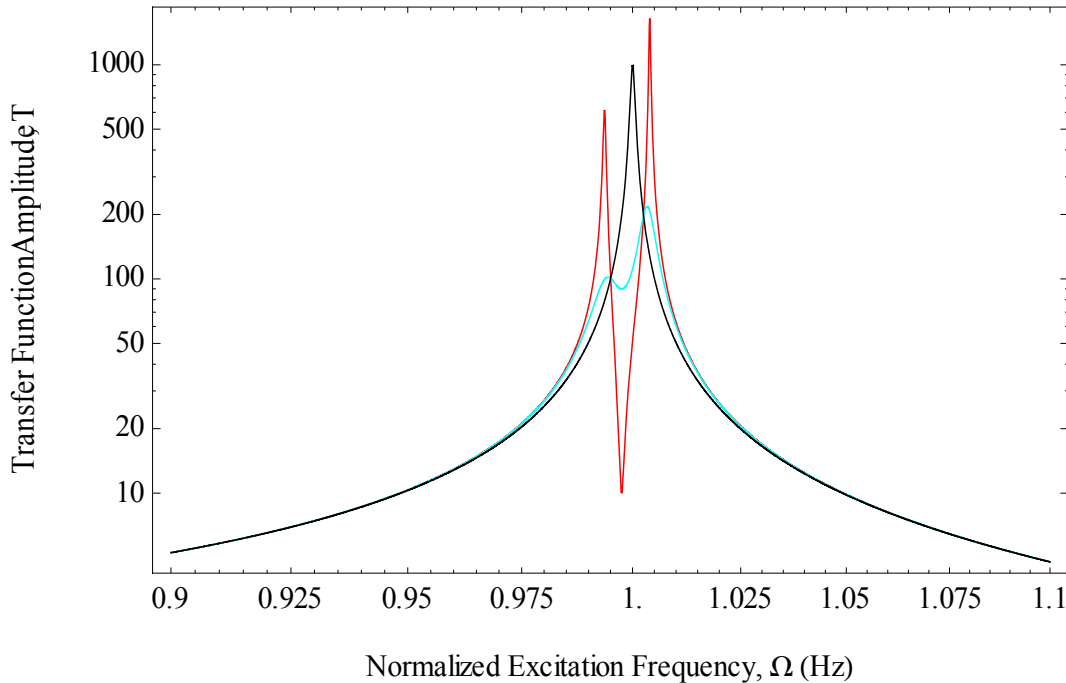

Figure 3: The Amplitude of the Transfer Function, T, as a function of the excitation frequency normalized by the primary mass, undamped resonant frequency, , for $\delta_1 = 0$, n = 0.9975 and $\mu = 10^{-4}$ and frequency independent moduli. Red curve for $\delta_2 = 0.001$, cyan curve for $\delta_2 = 0.01$ and black curve for $\delta_2 = 0.1$

3. Peak Amplitude of the Coupled System

For the case when the absorber damping is relatively high ($\delta_1 \approx 0.1$), the frequency at which the transfer function has a maximum corresponds to the real root of the denominator of the square of the transfer function:

$$\Omega^2_{peak} = root\left\{\left[\mu\Omega^4 - \Omega^2\left(1+n^2\right)+n^2\right]^2 + \left(n^2\delta_2\right)^2\left(1-\Omega^2\right)^2\right\}$$

The derivative of the denominator, with respect to Ω, is quite linear in the vicinity of Ω =1. Consequently we take the first term in a series expansion of the derivative of the denominator about 1 (i.e. Ω = 1 + ε) as an approximation of the denominator for root solution. The resulting root solution is further simplified by taking the first term in a series expansion in $r = \dfrac{M_2}{M_1}$ about

r = 0, since we are interested in small values of absorber mass relative to the modal mass. The resulting expression for the frequency at which the transfer function is a maximum is:

$$\Omega_{peak} = 1 + \frac{r\left(1-n^2\right)}{2 - 4n^2 + 2\left(1+\delta_2^2\right)n^4}$$

(see appendix A for the derivation).

In the limit as the absorber frequency is much different from the primary mass resonance (n >> 1 or n << 1) and/or the absorber damping is low ($\delta_2 \ll 0.1$), the apparent loss factor (inverse of the Q at resonance) will be lower than the actual test mass loss factor, due to the assumption that the test mass loss factor was negligible. The Q at resonance can be limited by the test mass loss factor in the following manner:

$$Q_p = \frac{1}{\sqrt{\dfrac{1}{T_{peak}^2} + \delta_1}}$$

where, as a worse case, we assume that $\delta_1 \equiv 10^{-7}$

The amplitude of the primary mass resonance, as a function of absorber frequency, for the case where the absorber mass is equal to 10^{-4} times the primary mass (or ~1 gm for a 10 kg modal mass) for three values of absorber loss is shown in Figure 4.
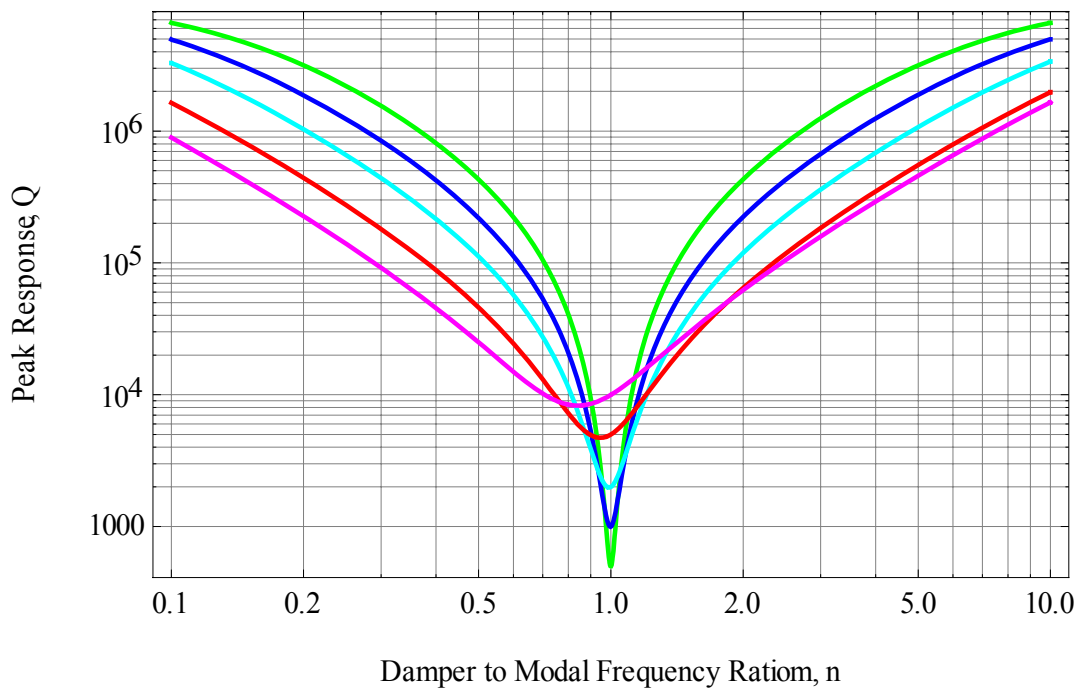

Figure 4: The Q of the primary mass resonance versus the ratio of the absorber resonant frequency to the primary mass, undamped frequency for an absorber to primary mass ratio of 10^{-4}. Green, blue, cyan, red and magenta curves are with $\delta_2 = 0.05, 0.1, 0.2, 0.5$ and 1.0, respectively.
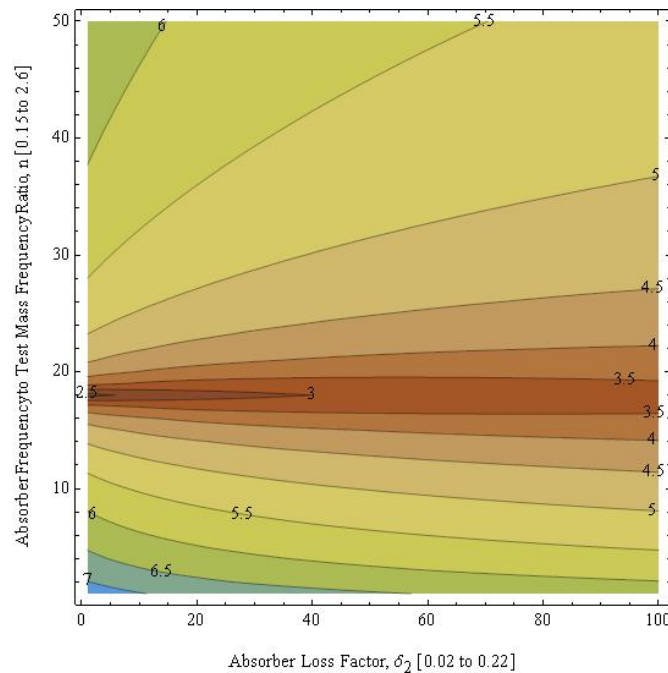
Figure 5: Contours of transfer function peak as absorber frequency and loss factor vary for an absorber mass of 10^{-4} times the test mass (contour values are Log_{10} of the Q)

In order to achieve a reduction in test mass Q from $\sim 10^{-7}$ to $\sim 10^{-5}$, one needs to have a (or several) well coupled (i.e. well placed) absorber(s) with a resonance within a factor of ~ 1.6 of each test mass resonance and a loss factor of ~ 0.1 (typical for passive, shunted piezoelectric dampers) or within a factor of ~ 3 and a loss factor of 1.0 (typical of high damping viscoelastic dampers).

The method to incorporate this analysis into an assessment of the performance of a chosen configuration of dynamic absorbers (number, position, frequencies, damping) based on test mass modal results is as follows:

1) For each dynamic absorber, i, and each test mass mode, j, determine the modal coupling (mode shape at the absorber location in the direction of the absorber degree of freedom), φ_{ji}.

2) Calculate the mass ratio for each absorber and test mass resonance pair, μ_{ji}, given the modal mass for resonance, M_j, the dynamic absorber mass, M_2, and the modal coupling, φ_{ji}.

3) Calculate the ratio of the absorber frequency to each test mass frequency, n_{ji}.

4) Calculate the peak response, Q_{pij}, from the above equation

4. Comparison with a Finite Element Simulation

4.1.   Simple 2-Degree of Freedom System

A model of the idealized coupled two-mass/spring system (Figure 1) was created with the Ansys finite element code[3]. The macro script used to create and solve the model is given in Appendix B. The transfer functions calculated for the three cases embodied in Figure 3, are given in Figure 6. There is a good match between the analytical formulation and the Ansys results.

4.2.   Test Mass and Idealized Damper

TBD – add an idealized mass-spring-damper to a node in a test mass finite element model and calculate the resulting transfer function change due to the presence of the dynamic absorber – compare to the formulation above.

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[3] Ansys release 11.0, Ansys Inc., www.ansys.com.

$\delta_2 = 0.001$

$\delta_2 = 0.01$

$\delta_2 = 0.1$

$\delta_1 = 10^{-7}$

n = 0.9975

ω_0 = 20 kHz
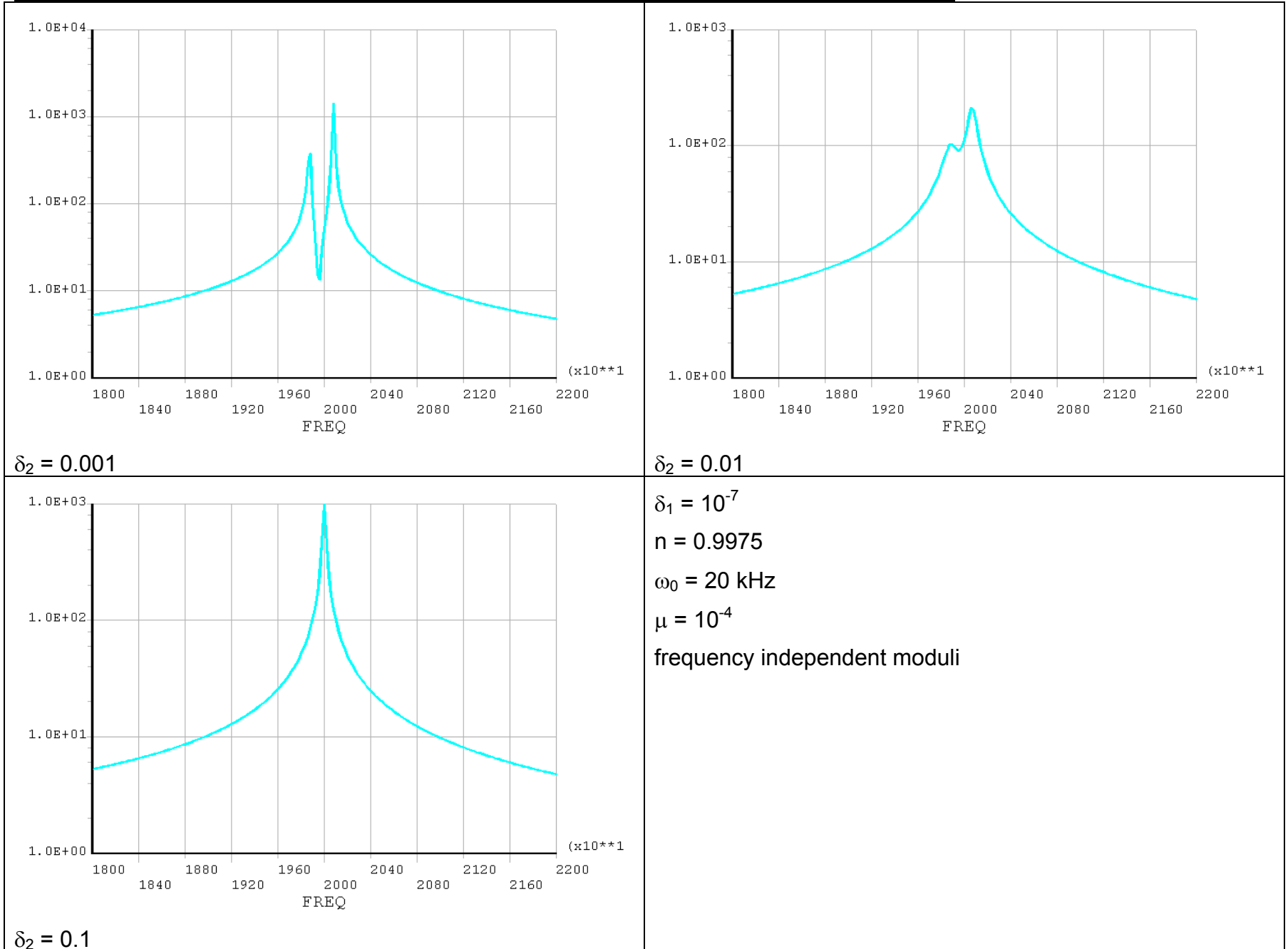
$\mu = 10^{-4}$

frequency independent moduli

Figure 6: Ansys transfer function calculation for the simple two degree of freedom system to compare analytical formulation results shown in Figure 3.

Appendix A: Derivation of the Dynamic Absorber Transfer Function

Appendix B: Ansys macro for the simple 2 DOF dynamic absorber

```
!**********************************************************************
! dynabs.mac
!**********************************************************************
! ANSYS macro to analyse transfer function for a dynamic absorber
! used as a check of the analytical formulation in T080194-01
!
! version 1
!Dennis Coyne 2008-09-05
!
finish
/CLEAR,START
/COM,ANSYS MODEL OF A SIMPLE TWO MASS-SPRING SYSTEM
/PREP7
/TITLE,DYNAMIC ABSORBER
!**********************************************************************
!*     PARAMETERS
!**********************************************************************
! values of parameters
! SI units (m,N,kg)

! large ground mass to represent a rigid connection, but allow for an applied
! ground acceleration (force) spectrum
m0=1e9

! undamped resonance
f0=20e3   ! Hz
w0 = 2*3.14*f0

! primary mass
m1=10   ! kg
k1=1.5791e11   ! N/m
g1=1e-7   ! structural damping ratio
c1=g1*k1/w0   ! effective viscous damping ratio at resonance
x1=1
!*
! secondary mass
m2=0.001   ! kg
k2=1.5713e7   ! N/m

! Change the structural damping ratio g2 = 0.1, 0.01, 0.001
! and compare to Figure 3 in T080194-01
g2=0.001   ! structural damping ratio
c2=g2*k2/w0   ! equivalent viscous damping near resonance
x2=1.1

!**********************************************************************
!*     GENERATE MODEL
!**********************************************************************
! ground
K,1,0,0,0,
!*
! primary mass
K,2,0,x1,0,
!*
! secondary mass
```

```
K,3,0,x2,0,

LSTR,1,2  ! primary spring
LSTR,2,3  ! secondary spring


!*********************************
!          Masses
!*********************************
ET,1,MASS21,0,0,0
TYPE,1     ! discrete mass elements
KEYOPT,1,3,2  ! translational dofs only
! ground mass
R,1,m0,m0,m0
REAL,1
KMESH,1
!       Primary Mass
R,2,m1,m1,m1
REAL,2
KMESH,2
!       Secondary Mass
R,3,m2,m2,m2
REAL,3
KMESH,3


!*********************************
!          Springs
!*********************************
ET,2,COMBIN14,1,0
TYPE,2       ! spring elements
KEYOPT,2,1,0  ! linear
KEYOPT,2,2,2  ! 1-D, UY dof
ESIZE,,1

R,4,k1,c1  ! primary spring
REAL,4
LMESH,1  ! primary spring

R,5,k2,c2  ! secondary spring
REAL,5
LMESH,2  ! secondary spring

NUMMRG,NODE,0.00001

!****************************************************************
!*      BOUNDARY CONDITIONS
!****************************************************************
!*
! Clamp the ground node
!DK, 1, ALL,0
!DK, 2, UX,0,UZ
!DK, 3, UX,0,UZ
D,ALL,UX,0
D,ALL,UZ,0

EPLOT

FINISH
```

```
!****************************************************************
!   Harmonic Analysis
!****************************************************************
/SOLU
ANTYPE,HARMIC
HROUT,OFF
LUMPM,ON
!EQSLV,SPARSE
F,1,FY,1e9
NSUBST,200
KBC,0
HARFRQ,18000,22000
OUTRES,ALL,ALL
SAVE
SOLVE
FINISH
/POST26
NUMVAR,200
NSOL,2,1,U,Y,GROUND
NSOL,3,2,U,Y,PRIMARY_MASS
NSOL,4,3,U,Y,SECONDARY_MASS
QUOT,5,3,2,,T
PLCPLX,0
/GROPT,LOGY,ON
PLVAR,5
EXTREM,5
/QUIT
```